Small Angle X-ray Scattering (I)

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Scattering at Low Angles

Inhomogeneous Density Distribution Over Large Distance (nm scale)

$s = 1/D$ (or $q = 2\pi / D$) ($Å^{-1}$)
$s = 0.001 - 0.1\ Å^{-1}$ D (10~ 1000 Å)
$2\theta = 0.008\text{-}8$, $\lambda = 1.542$ Å

$$s = |\bar{s}| = \frac{2\sin\theta}{\lambda}$$

- *Morphological information of multiphase system:*
  Domain (Particle) Size, Distribution, Surface Area, Interface Thickness
- *Density Fluctuation*
- *Supramolecular Ordered Structure (nanometer scale)*

Fundamental Theories

- X-ray scattering from the electron density distribution in sample
- Small angle scattering for the large distance

$$f(\vec{s}) = \int_{V_r} \rho(\vec{r}) e^{2\pi i \vec{r}\vec{s}} d\vec{r}$$

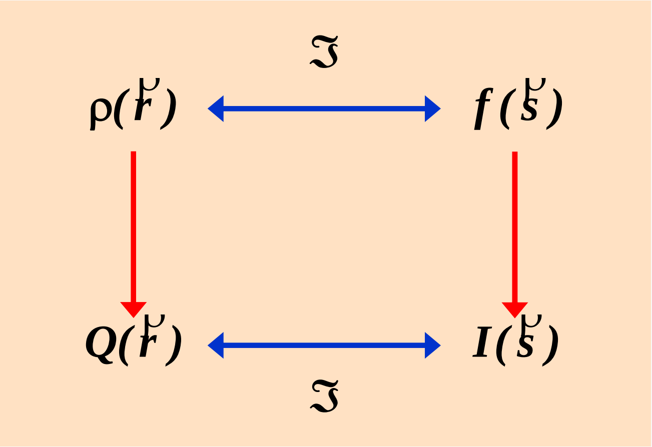
$$I(\vec{s}) = f(\vec{s}) \cdot f^{*}(\vec{s})$$

$$I(s) = \left|\Im\{\rho(x)\}\right|^{2}$$

$$\left|\bar{s}\right| = \frac{2\sin\theta}{\lambda} = \frac{1}{d} \qquad \left|\bar{q}\right| = \frac{4\pi\sin\theta}{\lambda}$$

$$I(\bar{s}) = \Im\left\{Q\left(\bar{r}\right)\right\}$$

$$Q(\bar{r}) = \Im^{-1}\left\{I\left(\bar{s}\right)\right\}$$



$f(\vec{s})$ **: Amplitude of scattered X-ray**

$I(\vec{s})$ **: Scattered intensity**

$\rho(\vec{r})$ **: Electron density function**

$Q(\vec{r})$ **: Patterson function** $(\rho(\vec{r}) * \rho(-\vec{r}))$

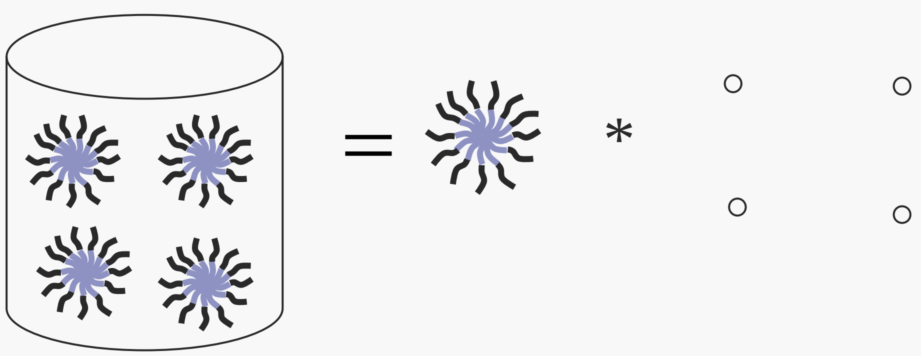
$\Im$ **: Fourier transform**

Convolution

$$\{f * g\}(\vec{r}) \equiv \int_{-\infty}^{\infty} f(\vec{u})\, g(\vec{r} - \vec{u})\, d\vec{u}$$

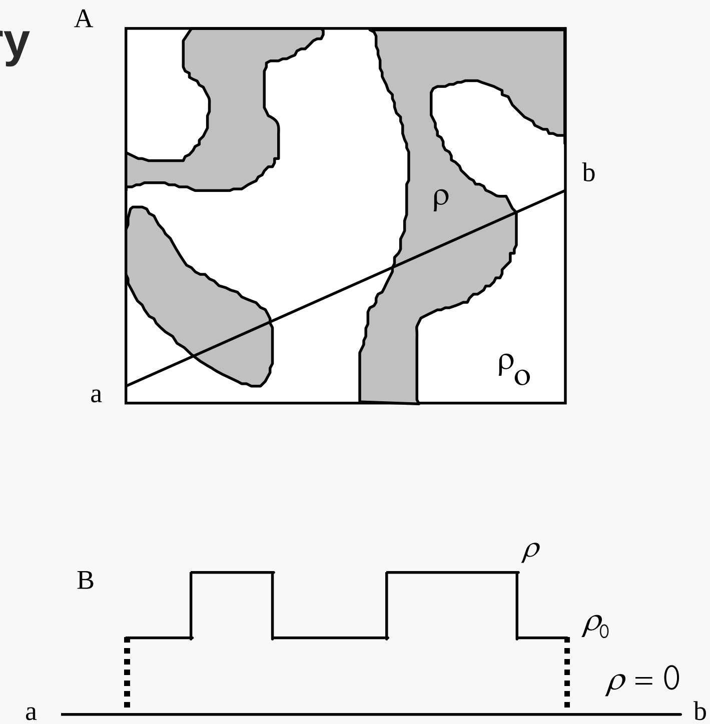
$$f * g(-\vec{r}) \equiv \int_{-\infty}^{\infty} f(\vec{u})\, g(\vec{r} + \vec{u})\, d\vec{u}$$

$$\Im\{f * g\} = \Im\{f\} \cdot \Im\{g\}$$



Two Phase System

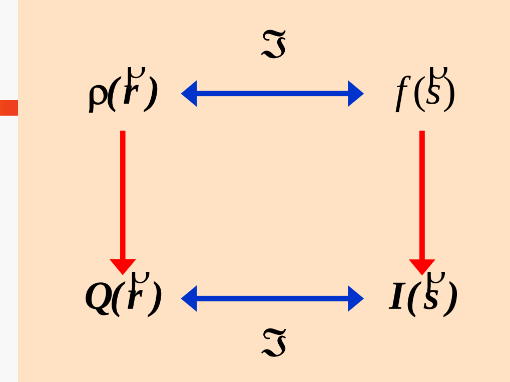
Sharp boundary



$\rho = 0$

Model of two phase system (A) and electron density distribution follow up line a-b (B).

Patterson Function



$$I(\bar{s}) = \Im\{Q(\bar{r})\} \qquad Q(\bar{r}) = \Im^{-1}\{I(\bar{s})\}$$

$$Q(\bar{r}) = \rho(\bar{r}) * \rho(-\bar{r}) = \int \rho(\bar{a})\rho(\bar{r}+\bar{a})\,d\bar{a}$$

$$Q(\bar{r}) = \int_0^{\infty} (\eta(\bar{a}) + \rho_0)(\eta(\bar{r}+\bar{a}) + \rho_0)\,d\bar{a} \qquad \eta(\bar{r}) = \rho(\bar{r}) - \rho_0$$

$$= \int \eta(\bar{a})\eta(\bar{r}+\bar{a})\,d\bar{a} + C$$

$$I(\vec{s}) = \Im\{\eta(\vec{r}) * \eta(-\vec{r})\} + \Im\{C\}$$

$$I_{obs}(\bar{s}) = \Im\{\eta(\bar{r}) * \eta(-\bar{r})\}$$

$$I_{obs}(\bar{s}) = \Im[Q_\eta(\bar{r})] \Leftrightarrow Q_\eta(\bar{r}) = \Im^{-1}\{I_{obs}(\bar{s})\}$$

Derivation

$$I(\bar{s}) = \Im\left\{\int \rho(\bar{u})\rho(\bar{u}+\bar{r})d\bar{u}\right\}$$

$$\eta(\bar{u}) = \rho(\bar{u}) - C$$

$$\int \rho(\bar{u})\rho(\bar{u}+\bar{r})d\bar{u} = \int\left[\eta(\bar{u})+C\right]\left[\eta(\bar{u}+\bar{r})+C\right]d\bar{u}$$

$$= \int \eta(\bar{u})\eta(\bar{u}+\bar{r})d\bar{u} + C\int \eta(\bar{u})d\bar{u} + C\int \eta(\bar{u}+\bar{r})d\bar{u} + C^2\int d\bar{u}$$

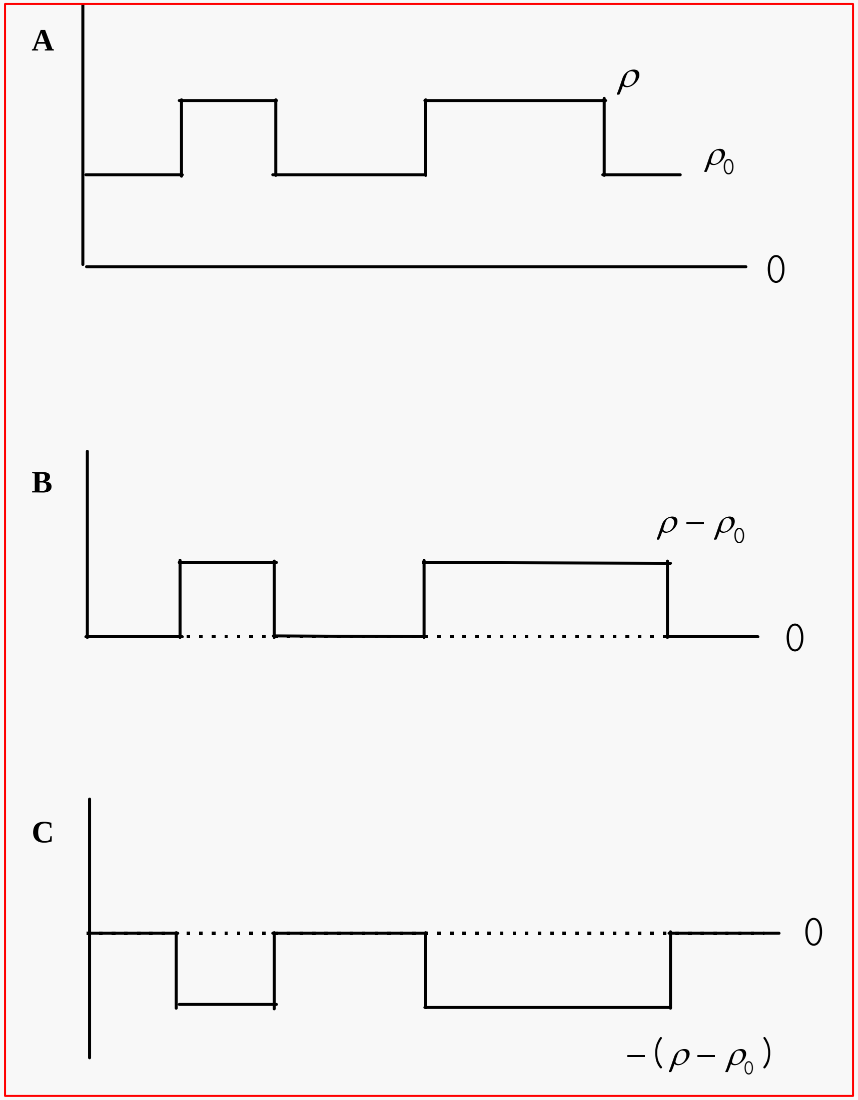
$$= \int \eta(\bar{u})\eta(\bar{u}+\bar{r})d\bar{u} + C'$$

$$I(\bar{s}) = \underline{\Im\left\{\int \eta(\bar{u})\eta(\bar{u}+\bar{r})d\bar{u}\right\}} + \underline{\Im\{C\}}$$

First term → I_{obs}

Second term → *delta function at s=0*

Babinet's Reciprocity Principle



All produces identical I_{obs}

Correlation Function and Patterson Function

$$\gamma(\bar{r}) = \frac{\eta(\bar{r}) * \eta(-\bar{r})}{\int_0^\infty [\eta(\bar{r})]^2 d\bar{r}} = \frac{\int_0^\infty \eta(\bar{a})\eta(\bar{r}+\bar{a})d\bar{a}}{\int_0^\infty \eta(\bar{a})\eta(\bar{a})d\bar{a}} = \Im^{-1}\{I_{obs}(\bar{s})\} \cdot \frac{1}{\langle \eta^2 \rangle V}$$

For an isotropic system $|\vec{s}| = s, \quad |\vec{r}| = r$

$$\gamma(r) = \Im^{-1}\{I_{obs}(s)\} \frac{1}{\langle \eta^2(u) \rangle V} = \frac{\int s^2 I_{obs}(s) \frac{\sin 2\pi rs}{2\pi rs} ds}{\int s^2 I_{obs}(s) ds}$$

Pair Distance Distribution Function (PDDF) $P(r) = r^2 \gamma(r)$

Invariant

Integration of Intensity

$$\int_0^\infty I_{obs}(\bar{s})d\bar{s} = \int_0^\infty I_{obs}(\bar{s})e^{2\pi i \bar{s}\bar{r}}d\bar{s}\,, \quad \bar{r} = 0$$

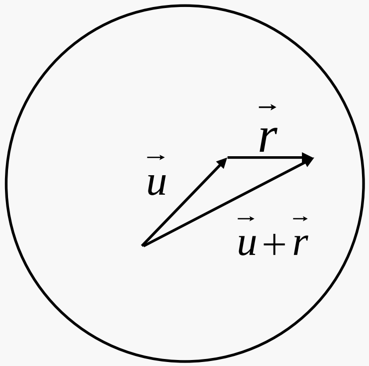
$$\mathfrak{I}^{-1}\left[I_{obs}(\bar{s})\right], \quad \bar{r} = 0$$

$$\langle \eta^2 \rangle V \cdot \gamma(0) = \langle \eta^2 \rangle V$$

Integration of intensity =
average density difference * scattering volume

Meaning of Correlation Function

$$\frac{\int_0^\infty \eta(\bar{a})\eta(\bar{r}+\bar{a})d\bar{a}}{\int_0^\infty \eta(\bar{a})\eta(\bar{a})d\bar{a}}$$



$$\rho(r)=\begin{Bmatrix}\rho & \text{inside} \\ 0 & \text{outside}\end{Bmatrix}$$

$\vec{u}$ ***and*** $\vec{u}+\vec{r}$ ***(inside the particle):*** $\gamma(\bar{r})=1$

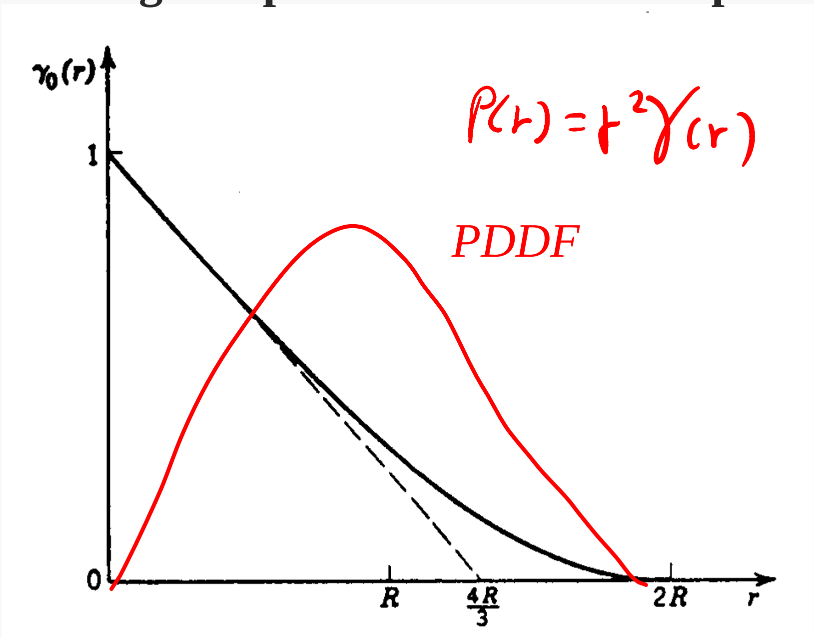
$\bar{u}+\bar{r}$ ***(outside the particle):*** $\gamma(\bar{r})=0$

γ depending on particle shape and size,
representing the probability of finding of a point u + r within the particle

Correlation function of sphere of radius R

$$\gamma(r)=0 \qquad r>2R$$

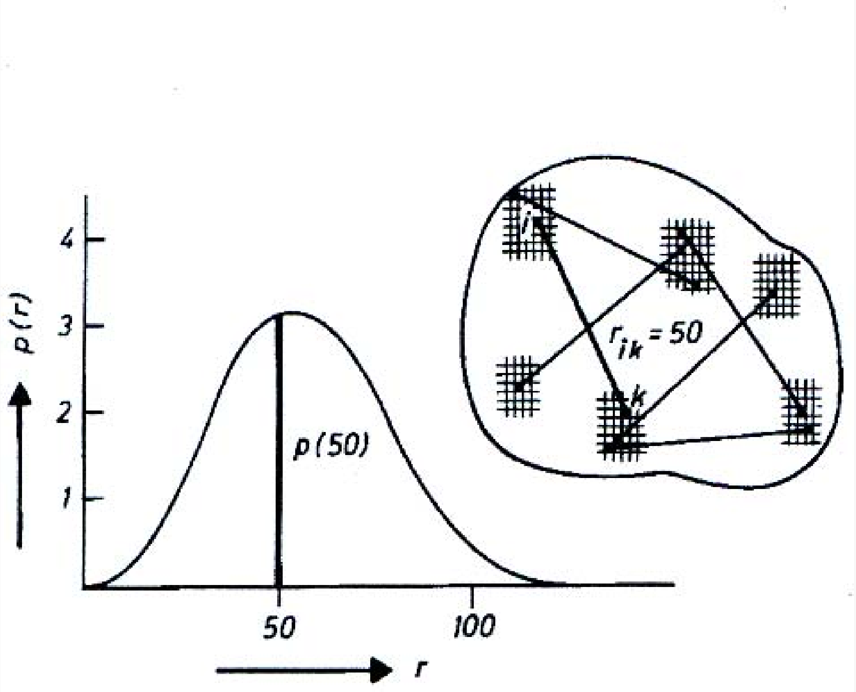
$$\gamma(r)=1-\frac{3}{4}\frac{r}{R}+\frac{1}{16}\left(\frac{r}{R}\right)^3$$



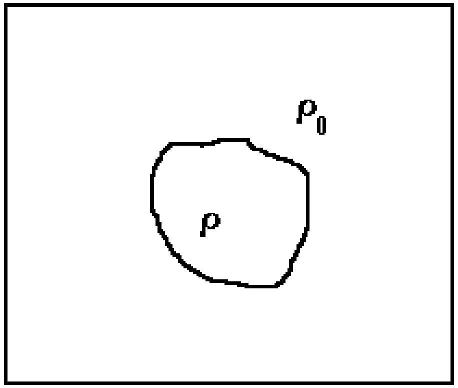
Pair Distance Distribution Function

$$P(r) = r^2\gamma(r)$$

Probability finding scattering elements separated by r



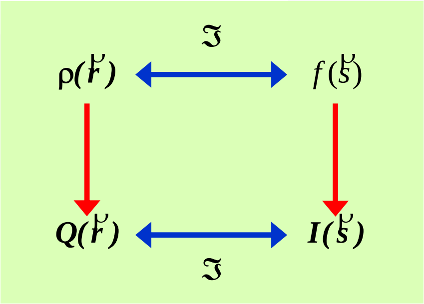
Particle Scattering (single particle)



$$\eta(\bar{r})\left(=\rho(\bar{r})-\rho_0\right)=(\rho-\rho_o)\,\sigma(\bar{r})$$

$$\sigma(\bar{r})\begin{Bmatrix}1 & inside \\ 0 & outside\end{Bmatrix}\quad form\ \ factor$$

$$f(s)=\Im\{(\rho-\rho_o)\sigma(r)\}=(\rho-\rho_o)F(s)$$

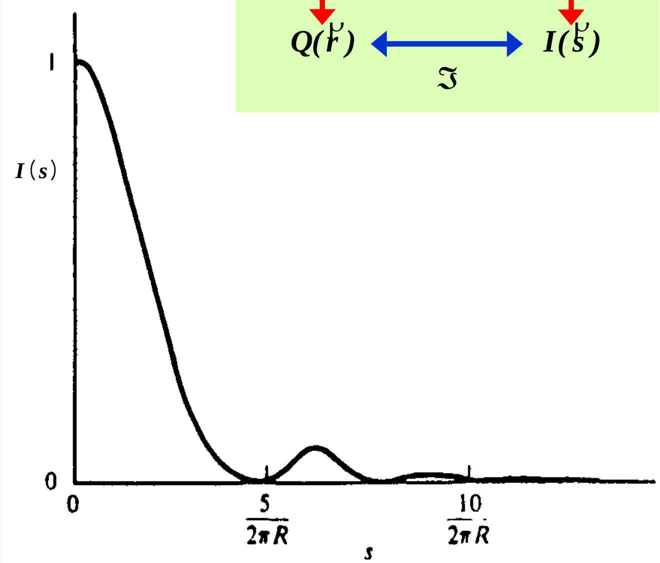


$$I_{obs}(\bar{s})=\Im\{\eta(\bar{r})*\eta(-\bar{r})\}=(\rho-\rho_0)^2\left|F(\bar{s})\right|^2$$

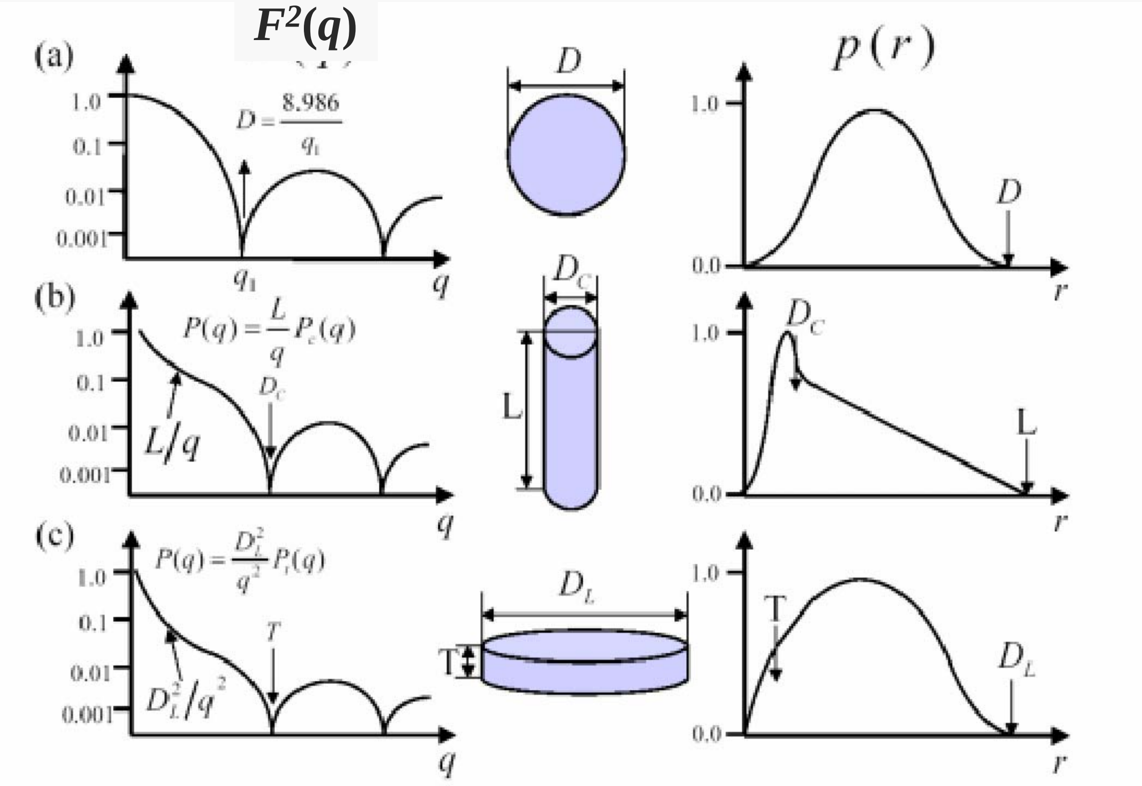
For Spherical Particles

$$I_{obs}(s)=$$

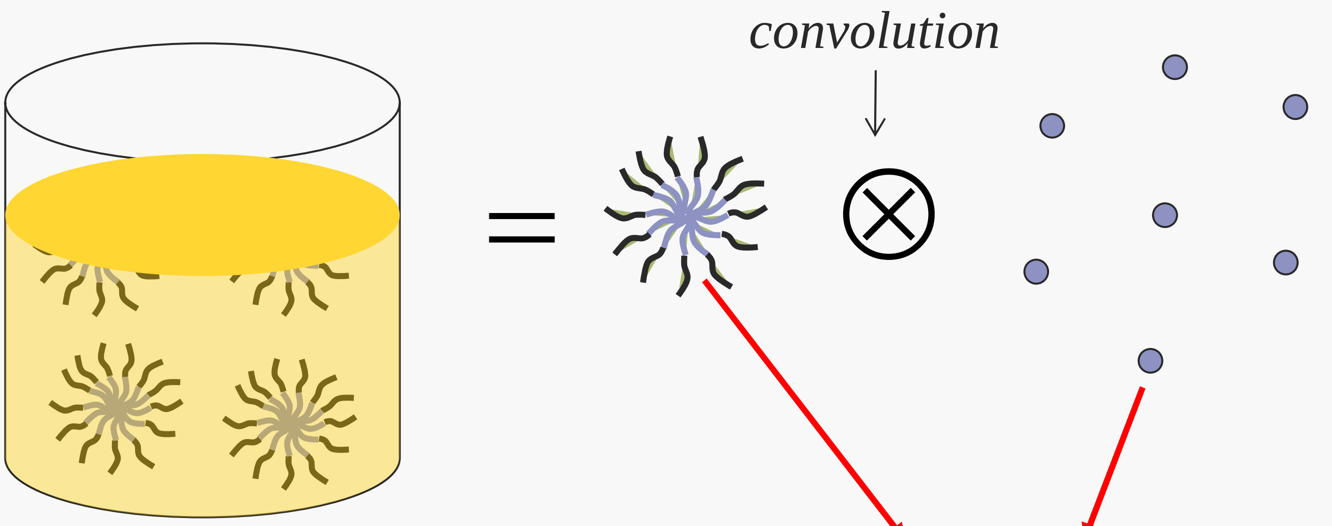
$$(\rho-\rho_0)^2\,\frac{4}{3}\pi R^3\left[3\,\frac{\sin(2\pi Rs)-2\pi rs\cos(2\pi Rs)}{(2\pi Rs)^2}\right]$$



Particle Scattering Pattern and PDDF

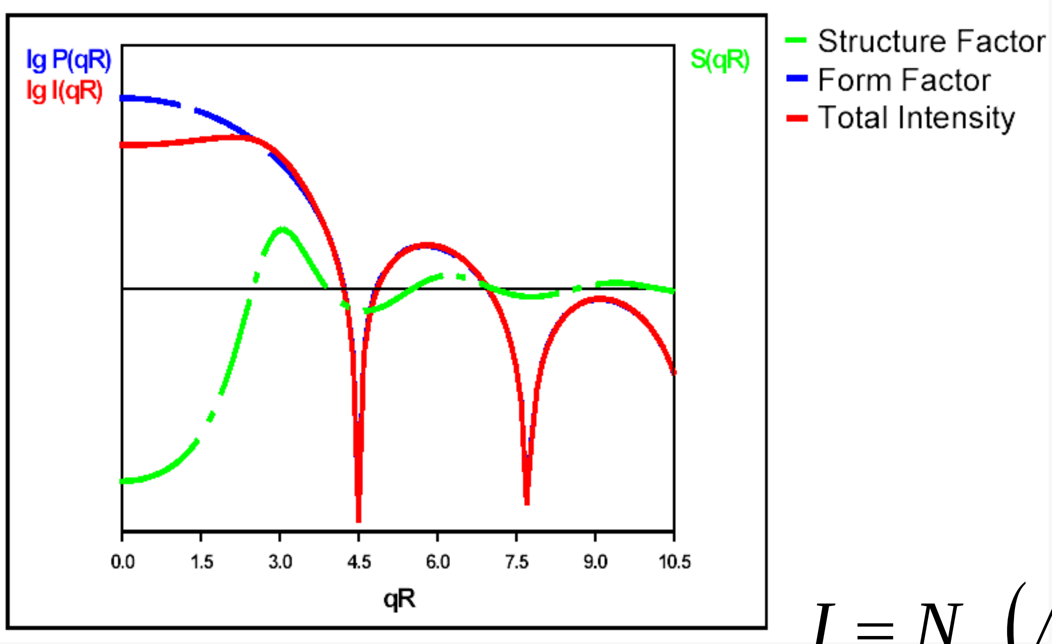


Particle Scattering



$$I = N_p \left(\Delta\rho_p\right)^2 \underbrace{F^2(q)}_{\text{Form Factor}} \underbrace{S^2(q)}_{\text{Structure Factor}}$$

$S(q) \approx 1$ for dilute solution



$$I = N_p \left(\Delta\rho_p\right)^2 \underbrace{F^2(q)}_{\text{Form Factor}} \underbrace{S^2(q)}_{\text{Structure Factor}}$$

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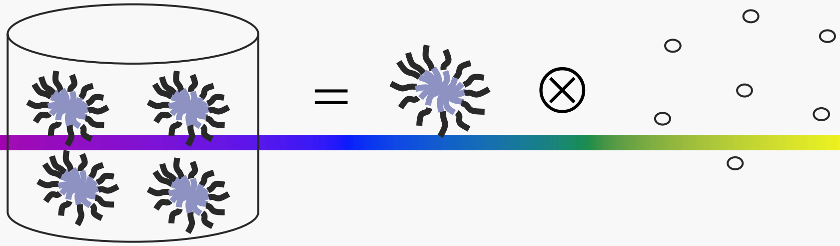
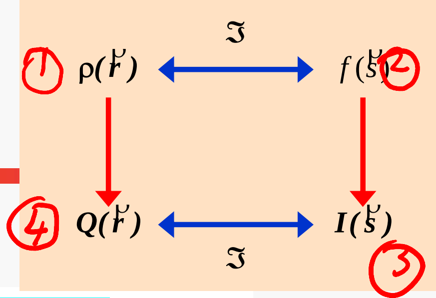
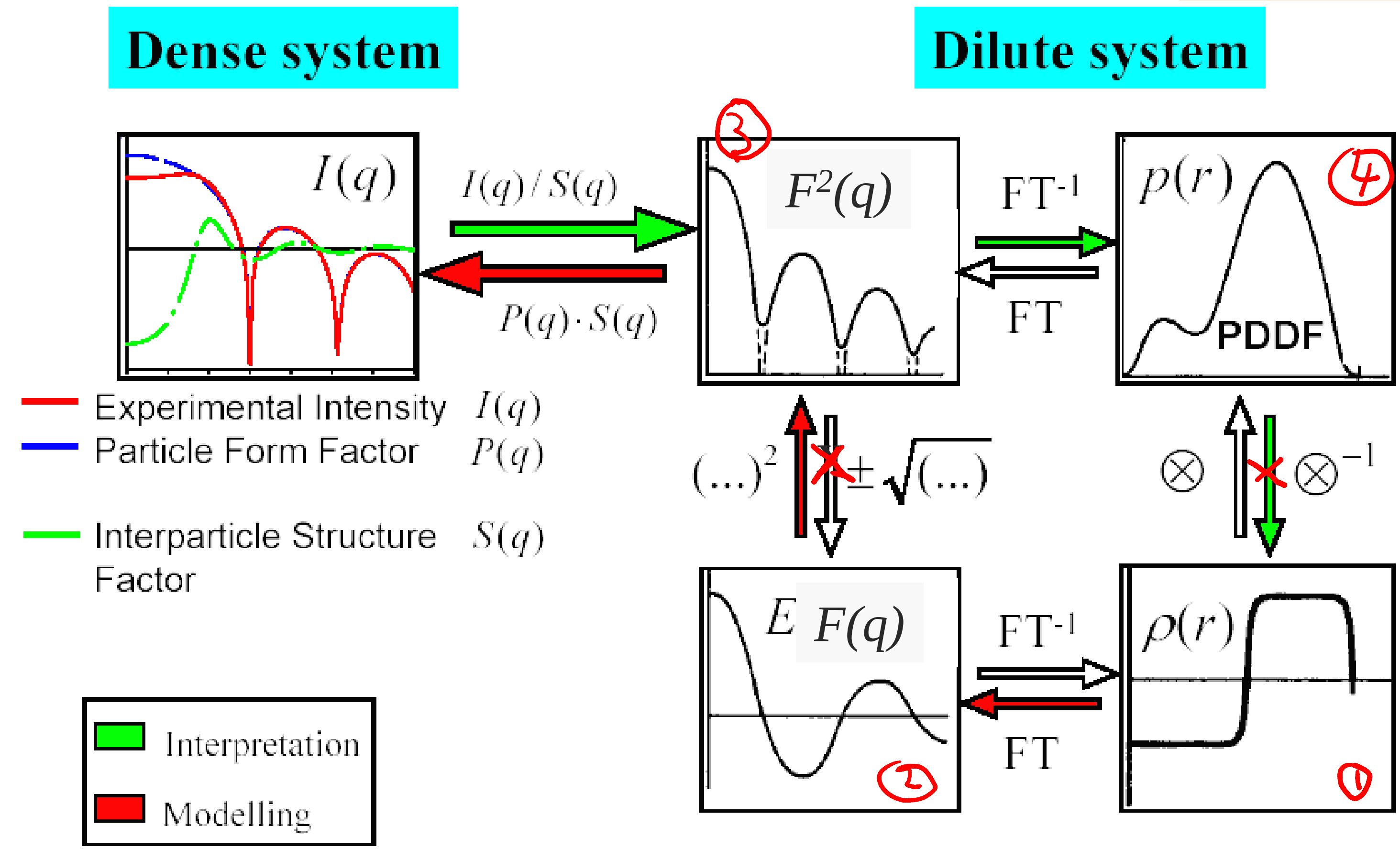
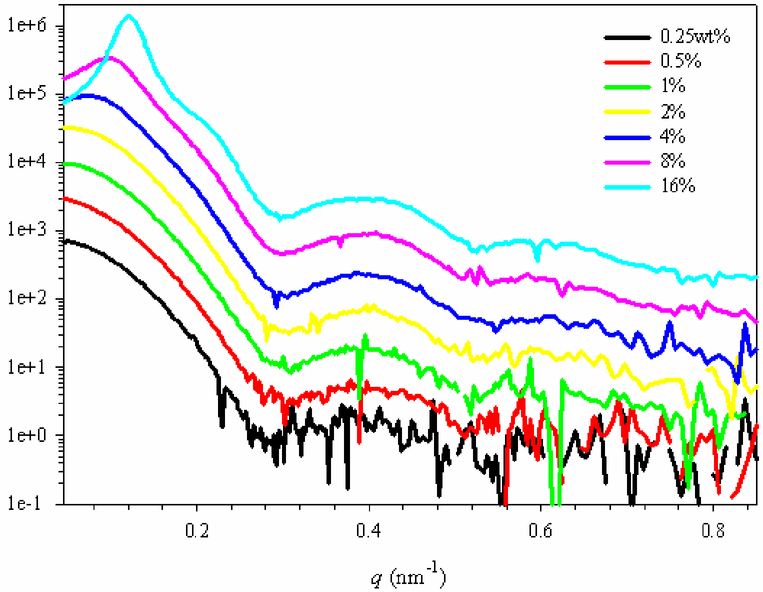
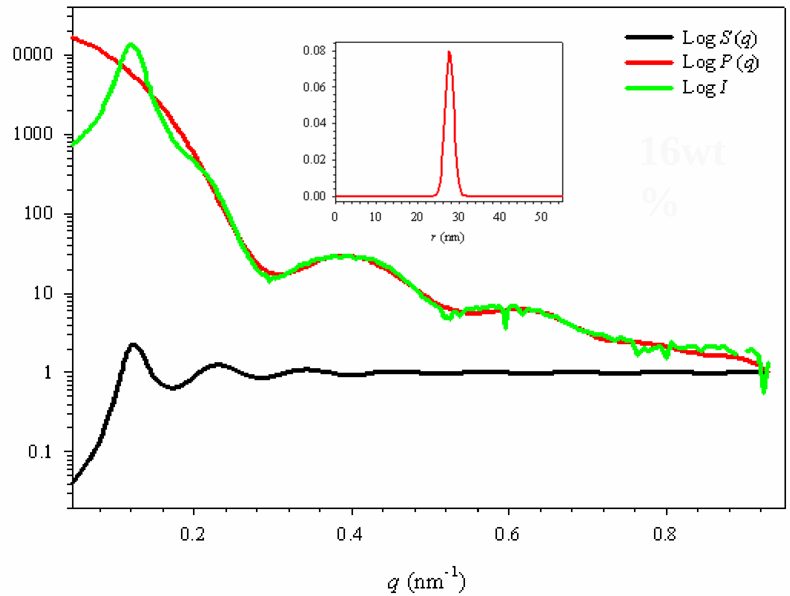

=
⊗

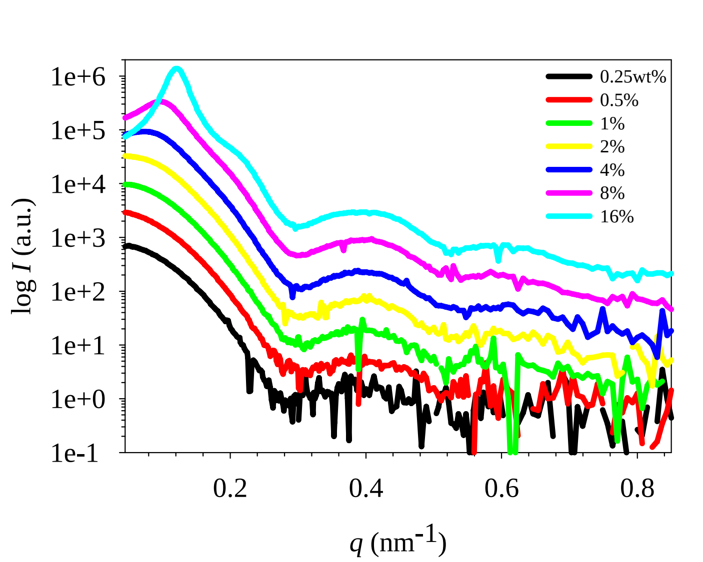

ℑ
ρ(r)
f(s)
Q(r)
I(s)
ℑ

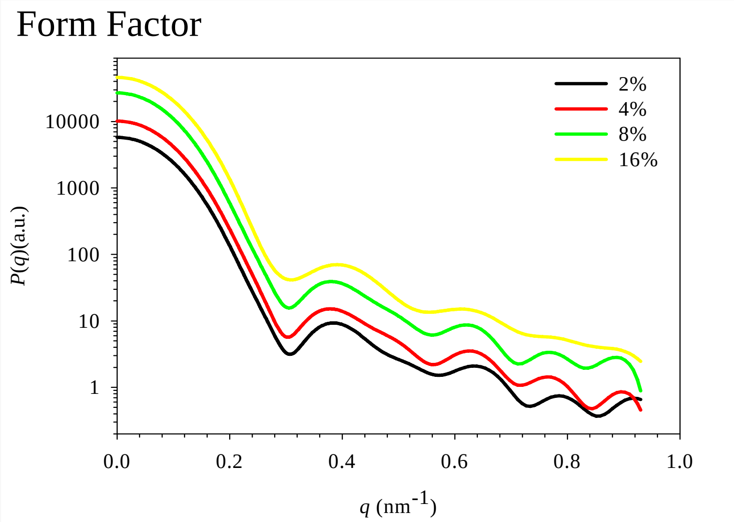

Dense system
Dilute system
I(q)
I(q) / S(q)
P(q) · S(q)
F²(q)
FT⁻¹
FT
p(r)
PDDF
Experimental Intensity I(q)
Particle Form Factor P(q)
Interparticle Structure Factor S(q)
(...)²
± √(...)
⊗
⊗⁻¹
E F(q)
FT⁻¹
FT
ρ(r)
Interpretation
Modelling

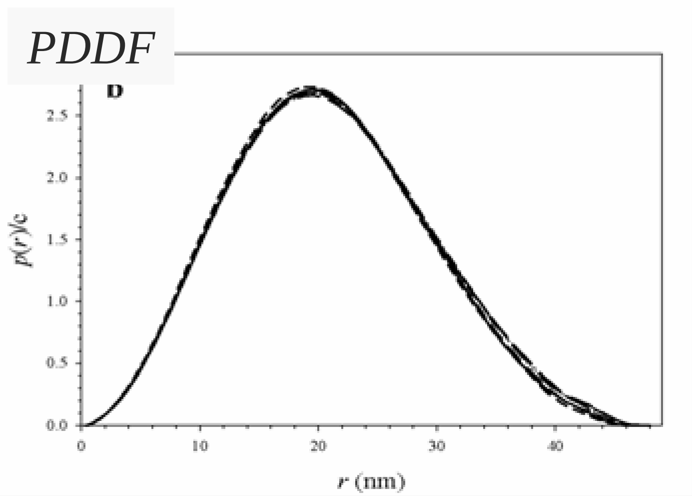
PS(12k)-*b*-PVP(11.8k) solutions in toluene



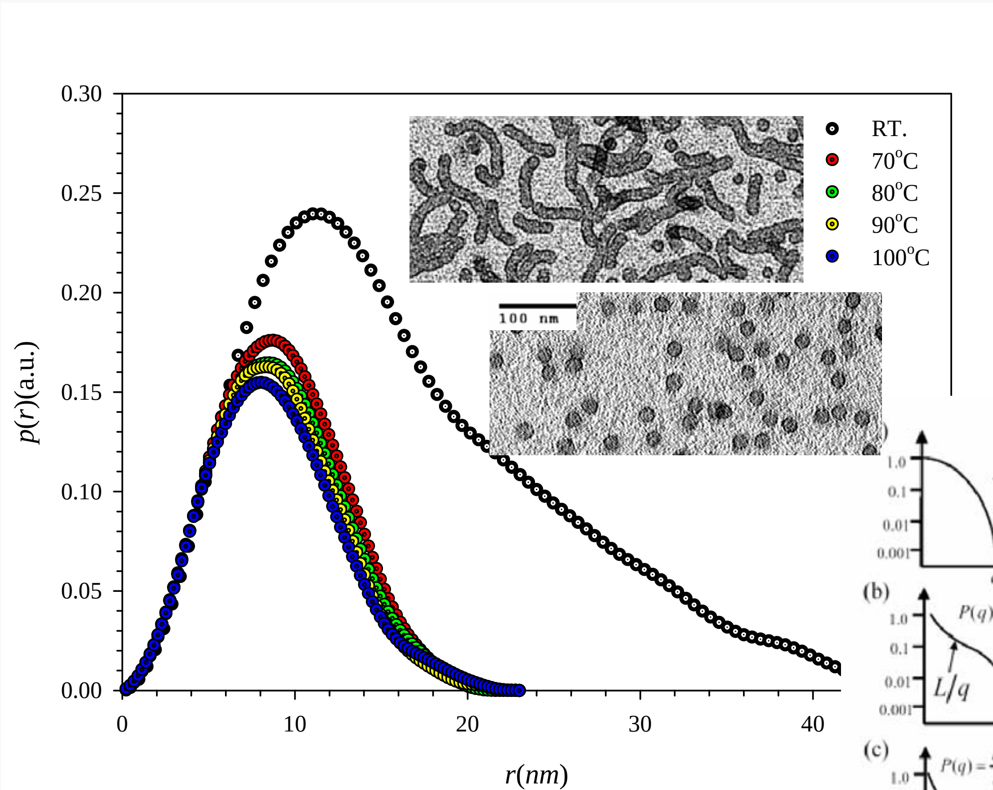




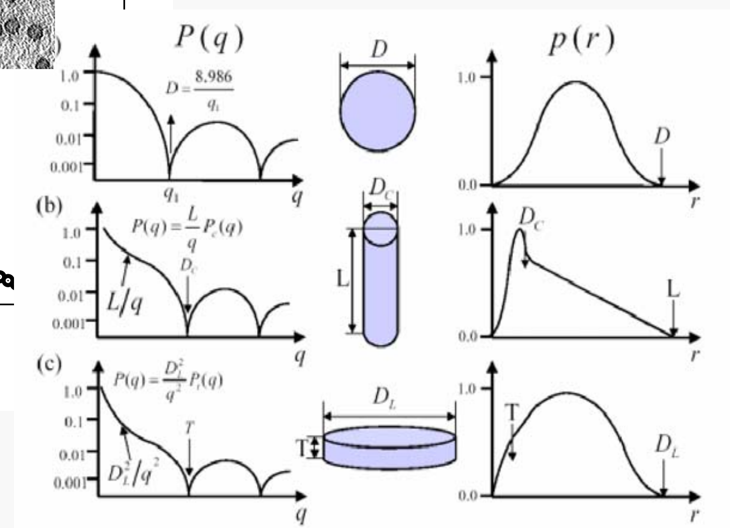




PDDF of PS(12K)-*b*-P4VP(11.8K), with different concentration levels up to 16 wt %.



p(r)(a.u.)
r(nm)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
10
20
30
40
RT.
70°C
80°C
90°C
100°C
100 nm



P(q)
p(r)
D
$D = \frac{8.986}{q_1}$
q_1
q
r
(b)
$P(q) = \frac{L}{q} P_c(q)$
L/q
D_C
L
(c)
$P(q) = \frac{D_L^2}{q^2} P_t(q)$
D_L^2/q^2
T
D_L
1.0
0.1
0.01
0.001
0.0

S.Y.Park

Guinier Approximation

$$I_{obs}(s) = N\langle\eta^2\rangle V^2 e^{-\frac{4}{3}\pi^2 R_g{}^2 s^2}$$

$$lnI_{obs}(s) = lnNV^2\langle\eta^2\rangle - \frac{4}{3}\pi^2 R_g{}^2 s^2$$

R_g : radius of gyration

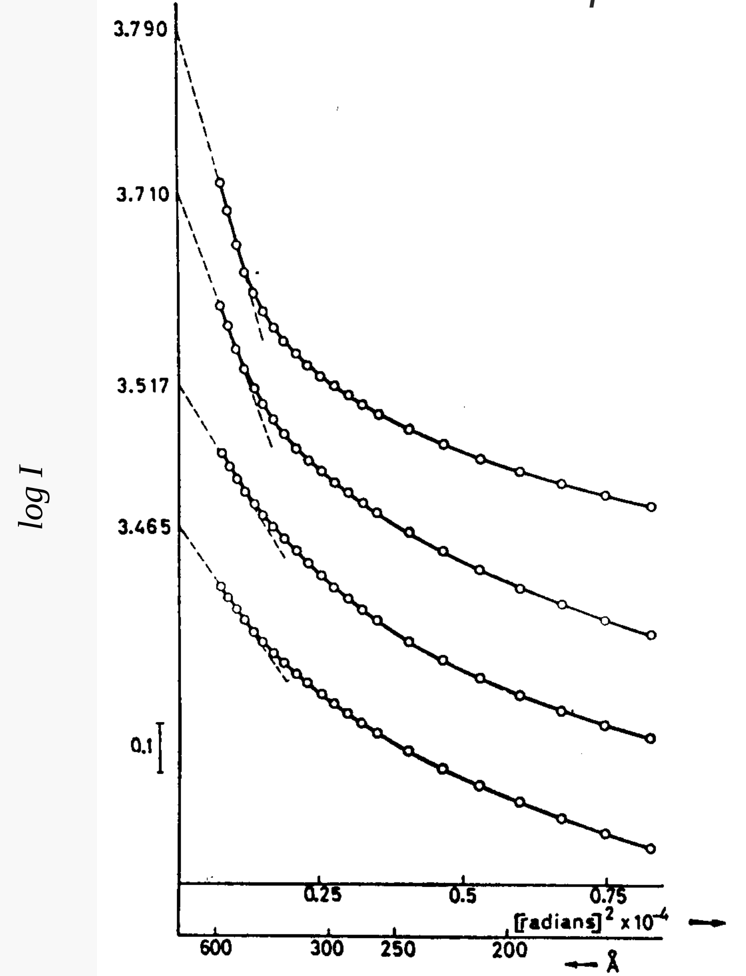
V : scattering volume

R_g from the slope

$$I(0) = NV^2\langle\eta\rangle^2$$

- *Applicable only at very small angles*
- *Must be sufficiently dilute*

Guinier plot of cellulose nitrate in acetone solution of 0.5 %.



Porod Law (for high scattering angle, Porod region)

For spherical particle

$$I_{obs}(s) = \frac{\rho - \rho_0}{8\pi^3}\left[\frac{4\pi R^2}{s^4} + \frac{1}{\pi s^6} + \frac{4R}{s^5}\sin 4\pi Rs + \left(\frac{4\pi a^2}{s^4} - \frac{1}{\pi s^6}\right)\cos 4\pi Rs\right]$$

when s is large

$$I_{obs}(s) = \frac{\rho - \rho_0}{8\pi^3}\frac{A}{s^4}$$

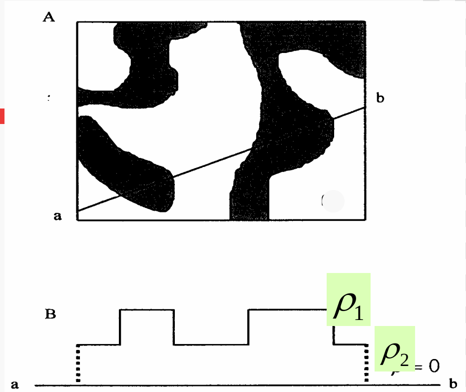
A is surface area of the particle

Satisfies regardless of particle shape, size and concentration

Surface area A can be obtained from the plot of $s^4 I_{obs}(s)\ vs.\ s$

This is also used for intensity fit at high angles

Condensed Multi-phase



ρ_1, ρ_2 : *densities of particle and matrix*

ϕ_1, ϕ_2 : *volume fractions* $\quad \rho_0 = \rho_1\phi_1 + \rho_2\phi_2$ *average density*

$$\eta(r) = \rho(r) - \rho_o$$

$$\langle \eta^2 \rangle = \Delta\rho^2 \phi_1\phi_2 \qquad \Delta\rho = \rho_1 - \rho_2$$

$$I_{obs}(\vec{s}) = \langle \eta^2 \rangle V \Im\{\gamma(\vec{r})\} = \Delta\rho^2 \phi_2\phi_1 V \cdot \Im\{\gamma(\vec{r})\}$$

$$I_{obs}(s) = 2\pi(\Delta\rho)^2 \phi_1\phi_2 V \cdot \Im\{\gamma(r)\}$$

Invariant

$$\int I_{obs}(\vec{s})\,d\vec{s} = \int I_{obs}(s)\,e^{i2\pi \vec{s}\vec{r}}\,d\vec{s} \quad (\bar{r} = 0)$$

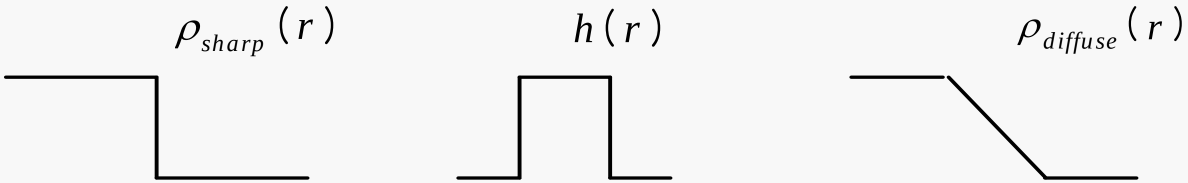
$$(\Delta\rho)^2 \phi_1\phi_2 V \gamma(\vec{r}), \quad \vec{r} = 0$$

$$(\Delta\rho)^2 \phi_1\phi_2 V$$

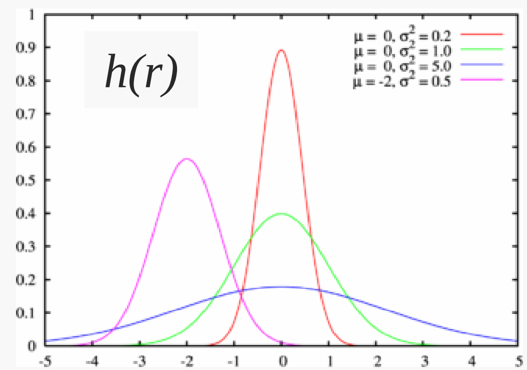
Surface Area

$$\gamma'(0) = -\frac{1}{4\phi_1\phi_2}\frac{A}{V}$$

Interface Thickness



$$\rho_{diffuse}(r) = \rho_{sharp}(r) * h(r)$$

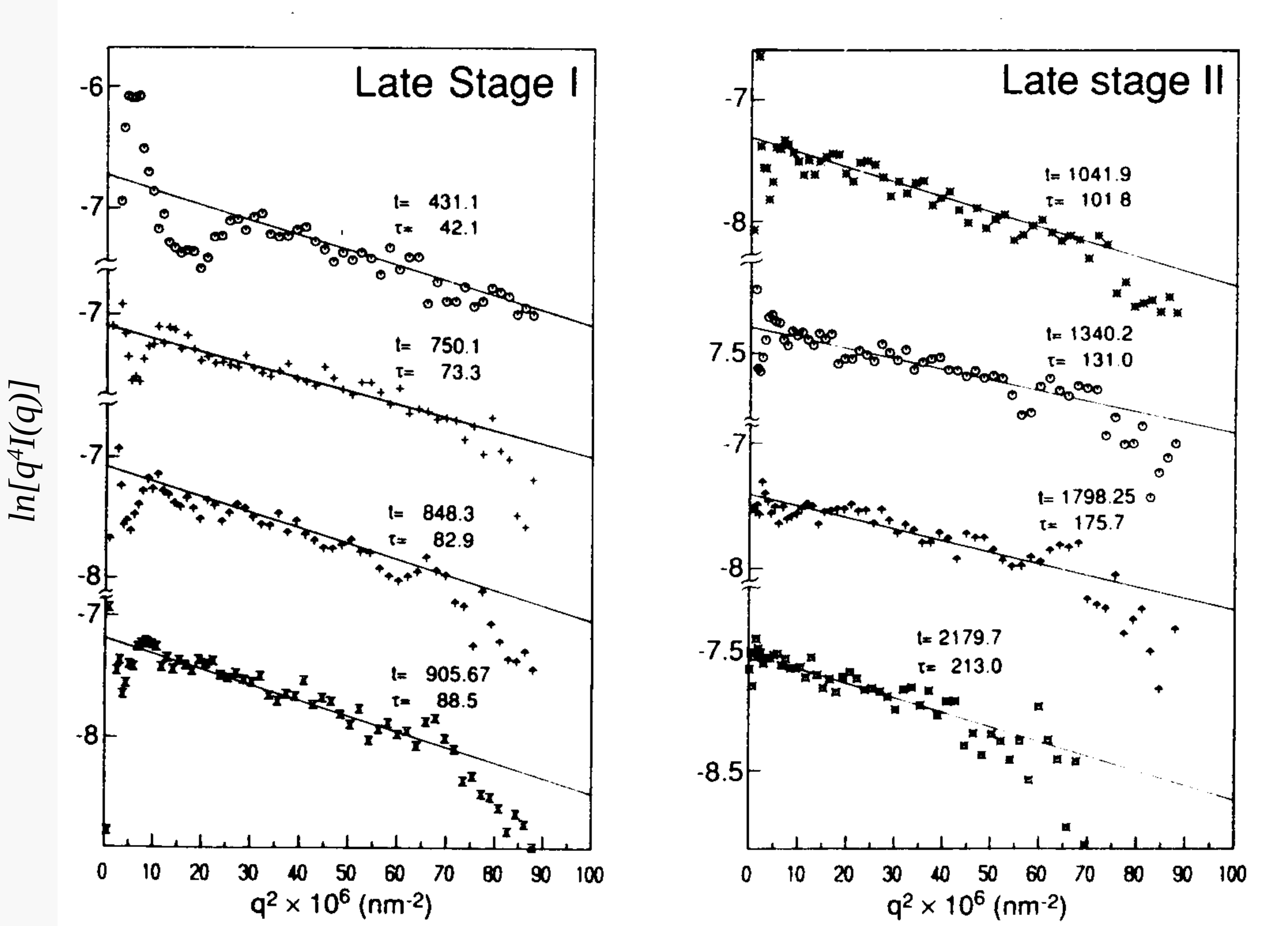


$$h(r) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^3} exp(-\frac{r^2}{2\sigma^2})$$

$$I_{diffuse}(\vec{s}) = I_{sharp}(\vec{s}) \left|\Im\left\{h(\vec{r})\right\}\right|^2 \qquad \left|\Im\left\{h(\vec{r})\right\}\right|^2 = e^{-4\pi^2\sigma^2 s^2}$$

$$I_{diffuse}(s) = I_{sharp}(s) e^{-4\pi^2\sigma^2 s^2} \sim I_{sharp}(s)(1-4\pi^2\sigma^2 s^2)$$

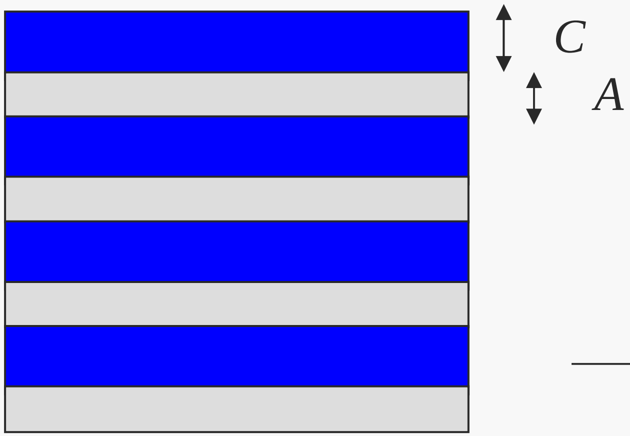
$$I_{diffuse}(s) \sim \frac{1}{8\pi}(\Delta\rho)^2 \frac{A}{s^4}(1-4\sigma^2\pi^2 s^2)$$ *Porod region*



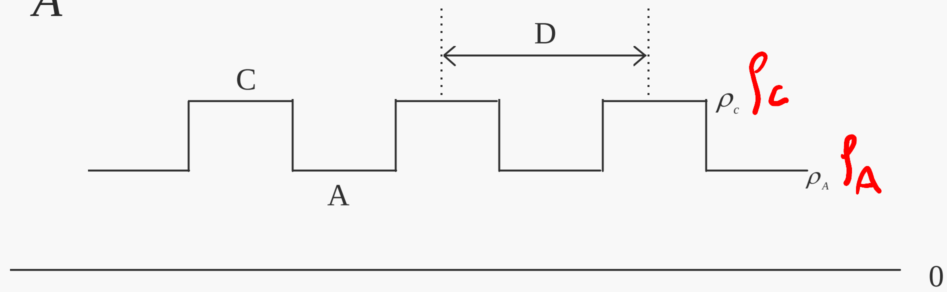
Late Stage I
Late stage II
ln[q⁴I(q)]
t= 431.1
τ= 42.1
t= 750.1
τ= 73.3
t= 848.3
τ= 82.9
t= 905.67
τ= 88.5
t= 1041.9
τ= 101.8
t= 1340.2
τ= 131.0
t= 1798.25
τ= 175.7
t= 2179.7
τ= 213.0
q² × 10⁶ (nm⁻²)

(a) (b)

Polymer Crystals (lamellae)

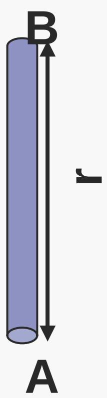


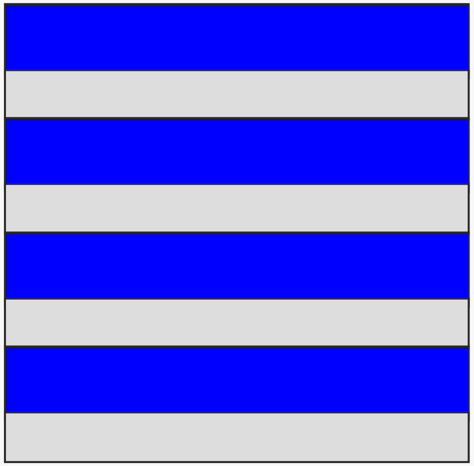
$$D = C + A$$



$$I_{1obs}(s) = 4\pi s^2 I_{obs}(s)$$

Correlation Function

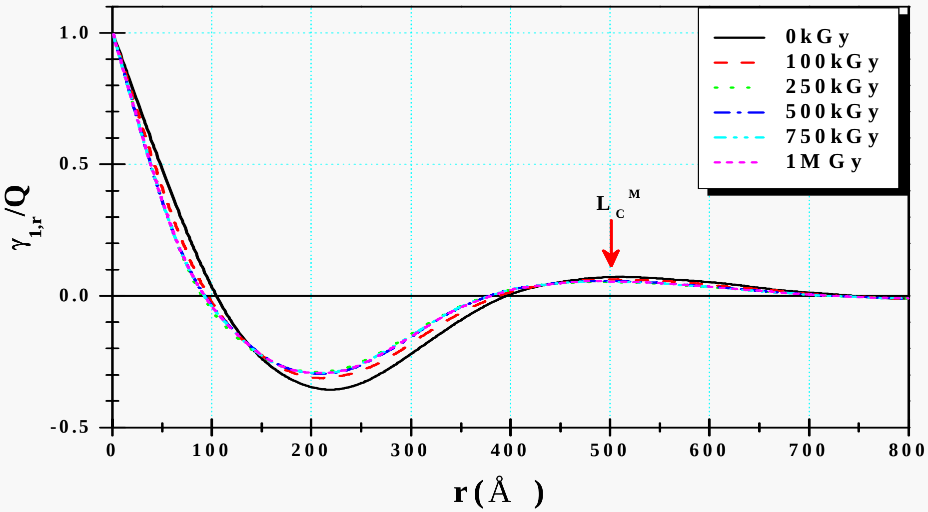




$$\eta_A,\ \eta_B\ \left(\eta = \rho - \langle\rho\rangle\right)$$

$$\gamma_1(r) = \frac{\int \eta(r+u)\,\eta(u)\,du}{\langle \eta^2 \rangle} = \frac{\langle \eta_A \eta_B \rangle}{\langle \eta^2 \rangle}$$

$$\gamma_1(r) = \frac{\int I_1(s)\cos 2\pi rs\,ds}{\int I_1(s)\,ds}$$



Correlation and Interface Distribution Functions Analysis on Lamellar Structure

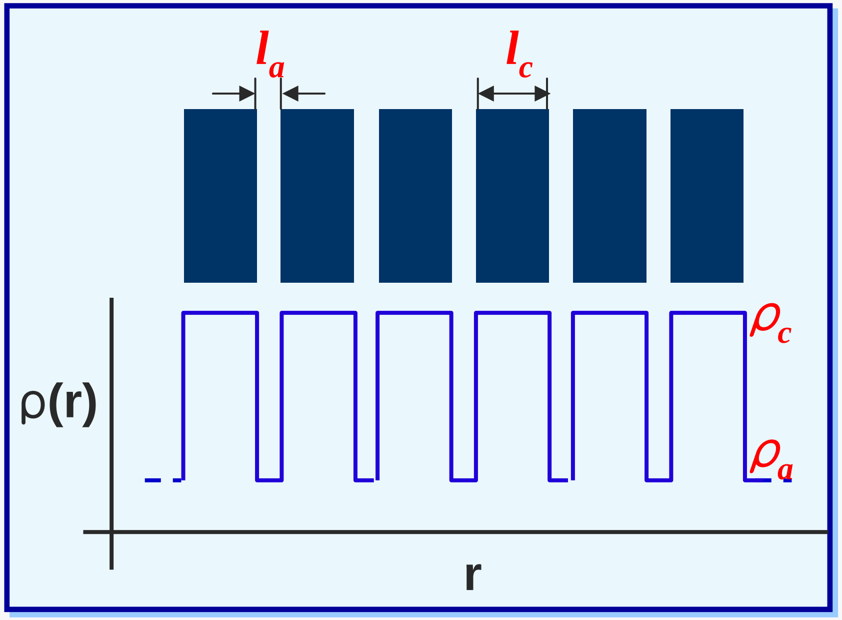
Examples of various arbitrary model

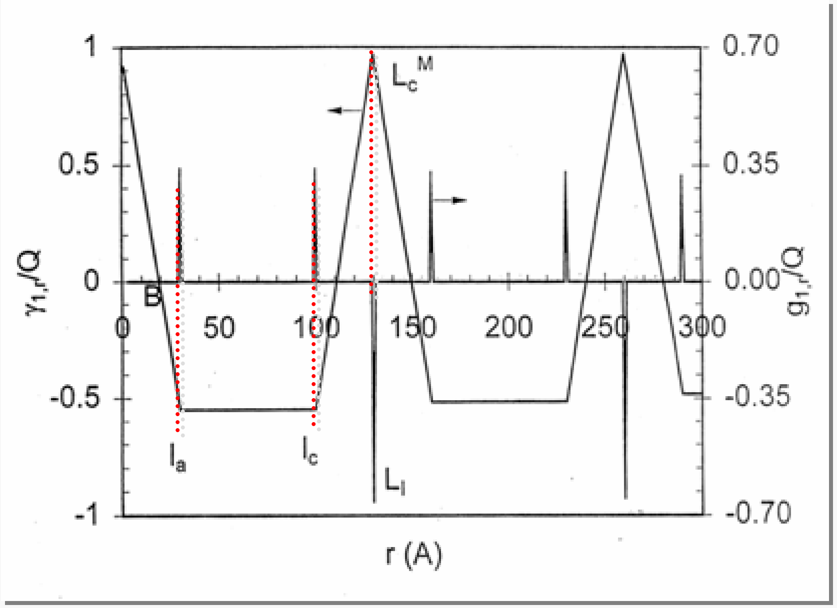
$$Z''(r) = \frac{2}{r_e^2 (2\pi)^2} \int_0^\infty \left[\lim_{q \to \infty} q^4 I(q) - q^4 I(q) \right] \cos qr dq$$

Ideal Two Phase Model

$$I_{ideal} = \Im(\rho_{ideal} * \rho_{ideal})$$

l_c=100Å, l_a=30Å, finite no. of lamellae in the stack is 20

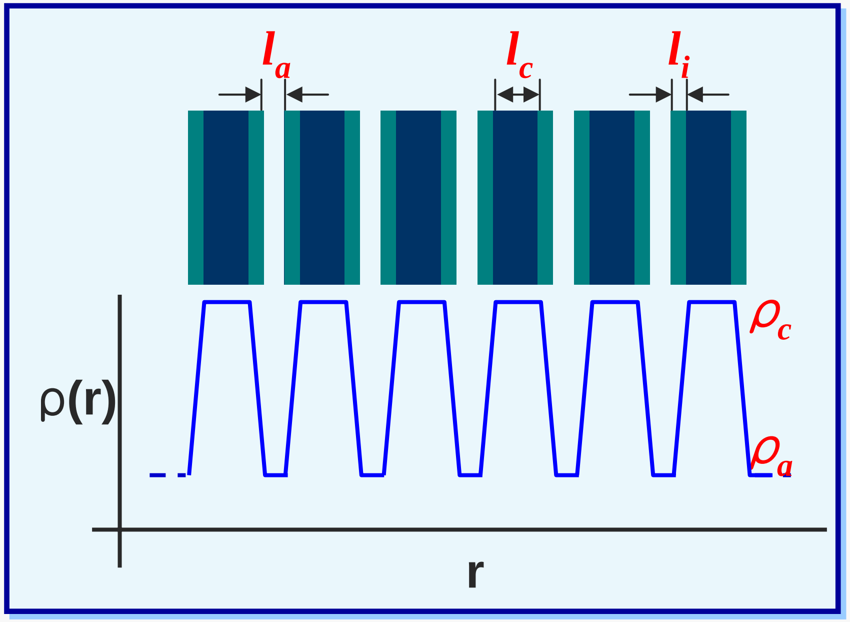


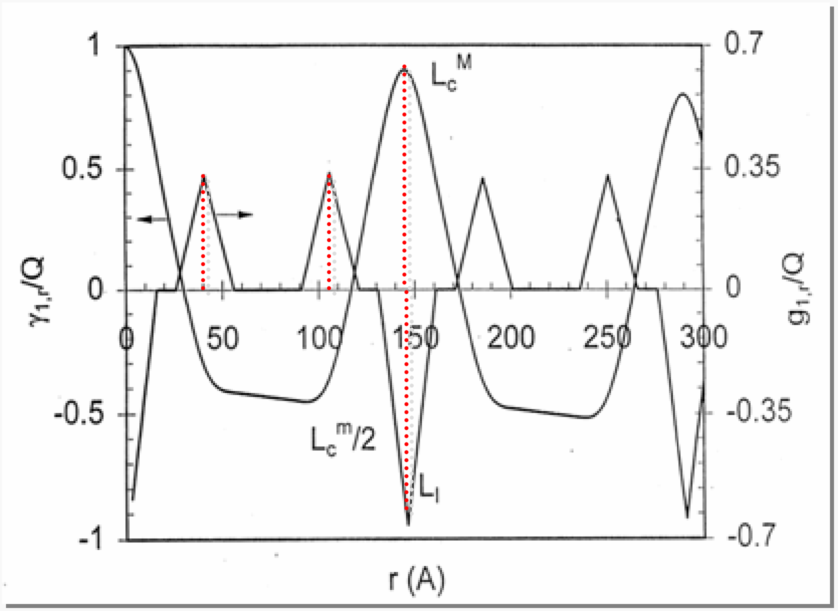


Model With Interface (I)

The ideal two phase model with a **finite crystal amorphous transition zone**

l_c=100Å, l_a=30Å, l_i=12Å

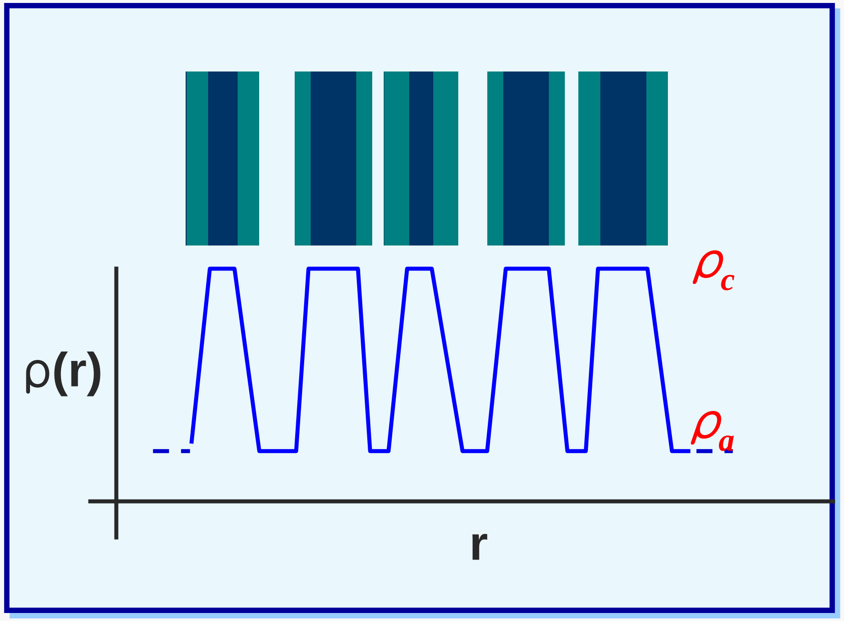


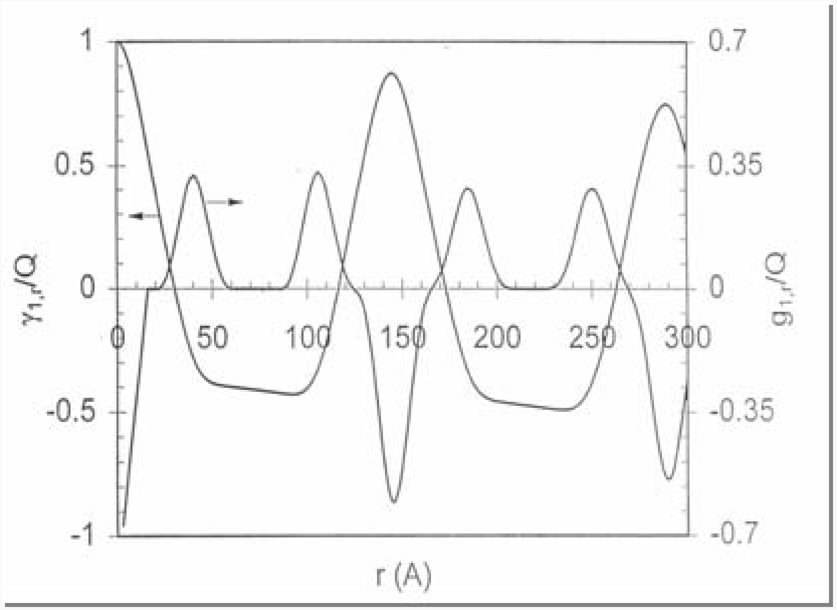


Model With Interface (II)

Distribution of lamellar and amorphous layer sizes

l_c=90Å, w_c=10Å, l_a=25Å, w_a=10Å, l_i=15Å, w_i=1Å





$$L_{model} = L_c^M \geq L_c^m \quad for\ w_2 \leq w_1$$

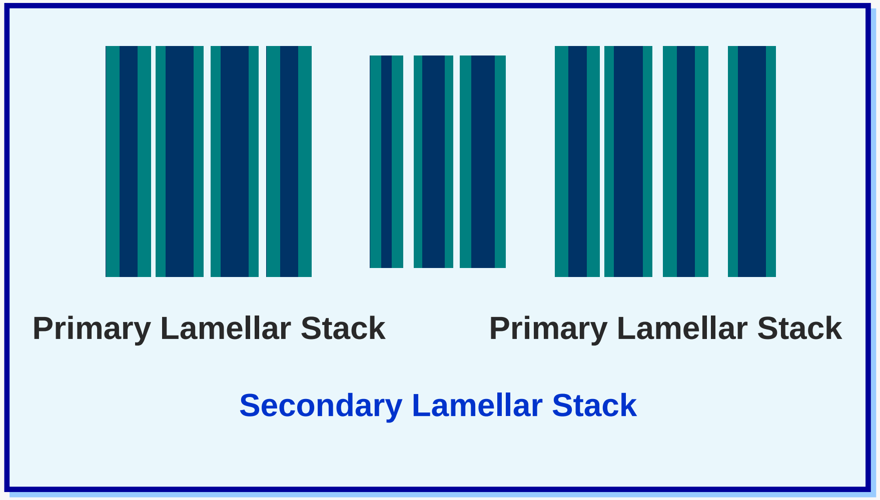
$$L_{model} = L_c^M \leq L_c^m \quad for\ w_2 \geq w_1$$

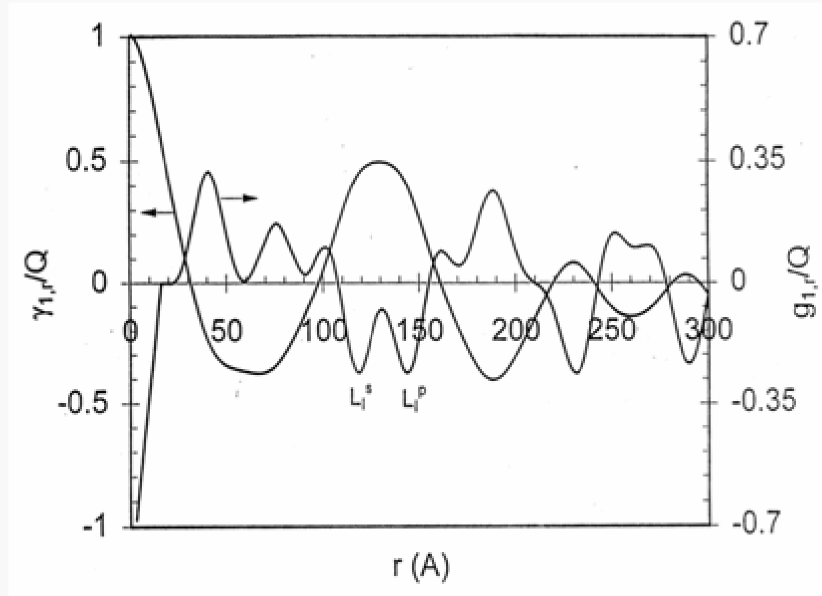
2 ; thicker phase
w; width of the thickness distribution
L_{model}; average long spacing in the model

Dual Lamellar Stack Model

Stack 1 l_c=90Å, w_c=10Å, l_a=25Å, w_a=10Å, l_i=15Å, w_i=1Å

Stack 2 l_c=60Å, w_c=10Å, l_a=25Å, w_a=10Å, l_i=15Å, w_i=1Å





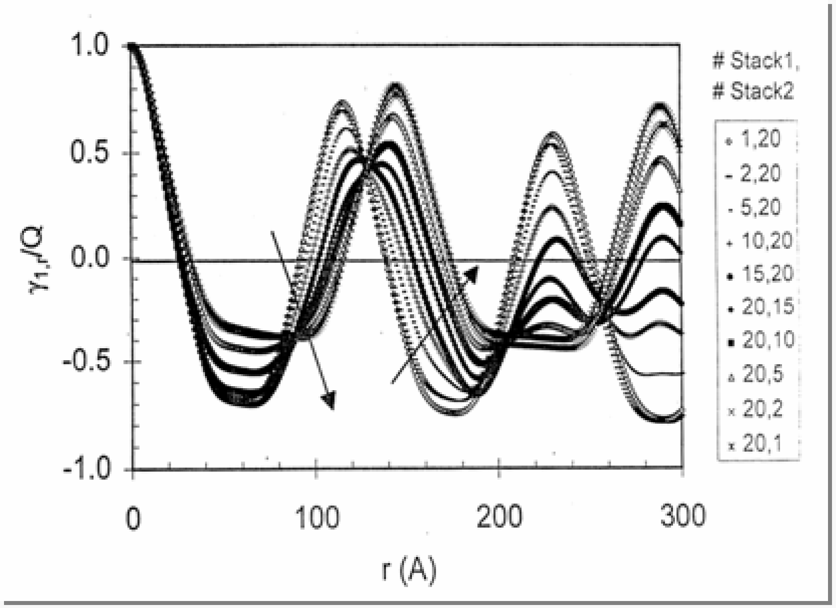
$$L_c^M \geq L_I\,,\quad L_c^m \geq L_I$$

$$L_c^M \geq L_c^m \geq L_I \quad for\ w_2 \leq w_1$$

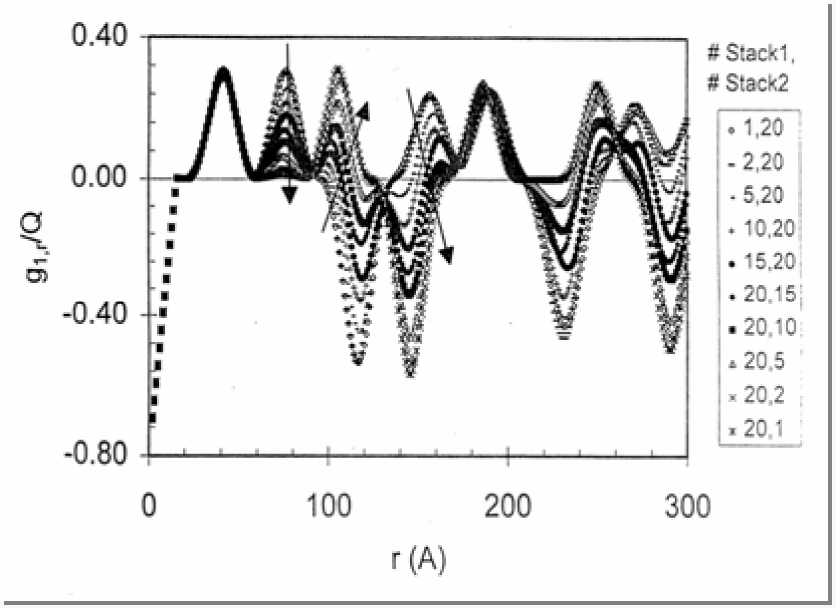
$$L_c^m \geq L_c^M \geq L_I \quad for\ w_2 \geq w_1$$

Examples

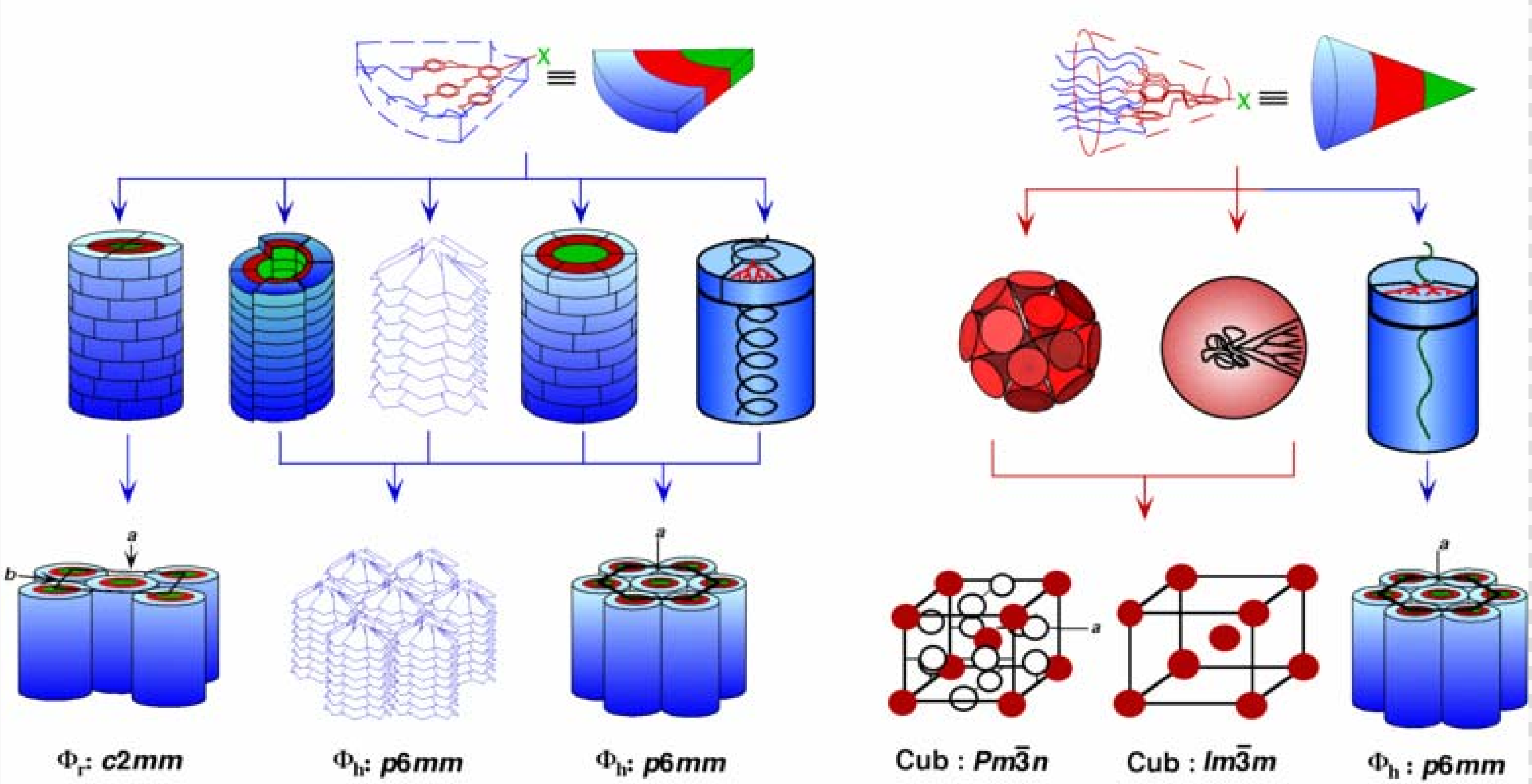
Correlation function



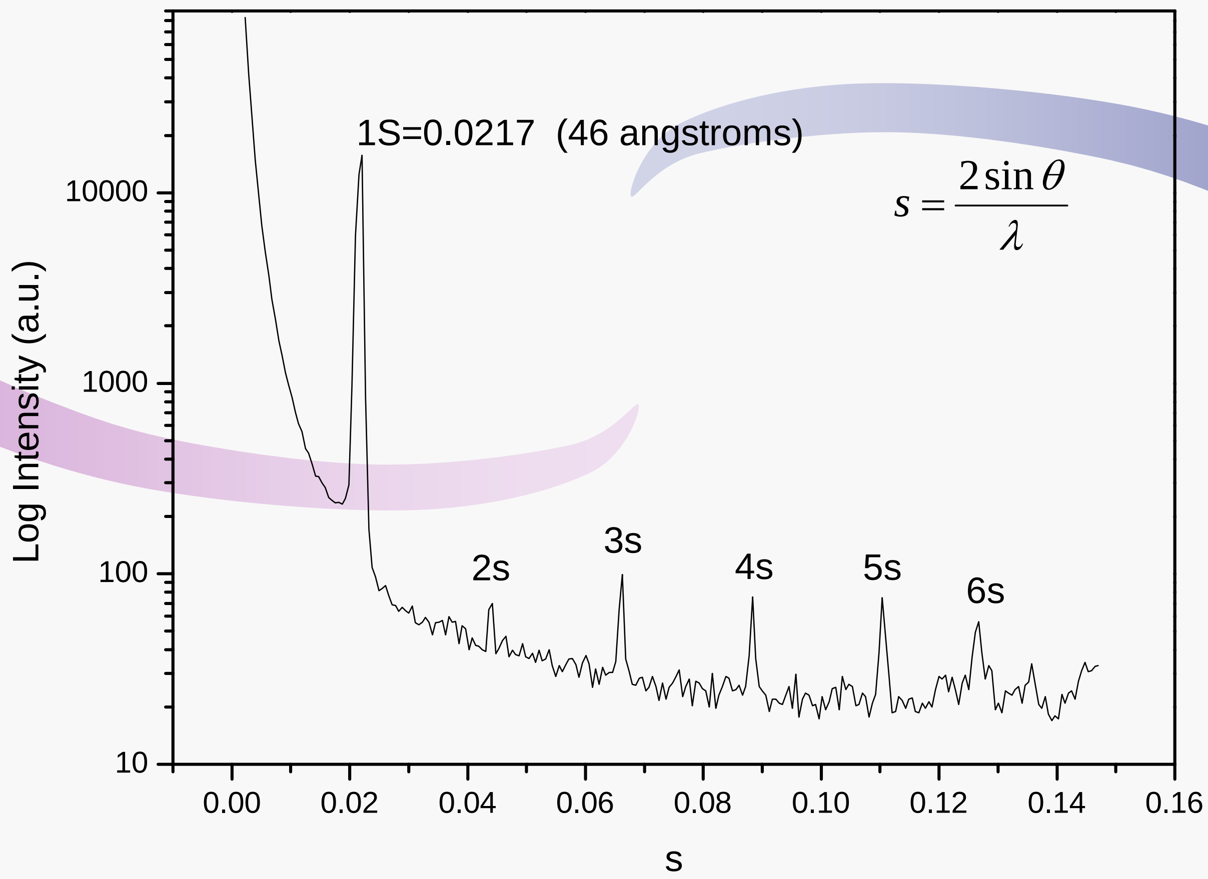
Interface distribution function



Nanostructures

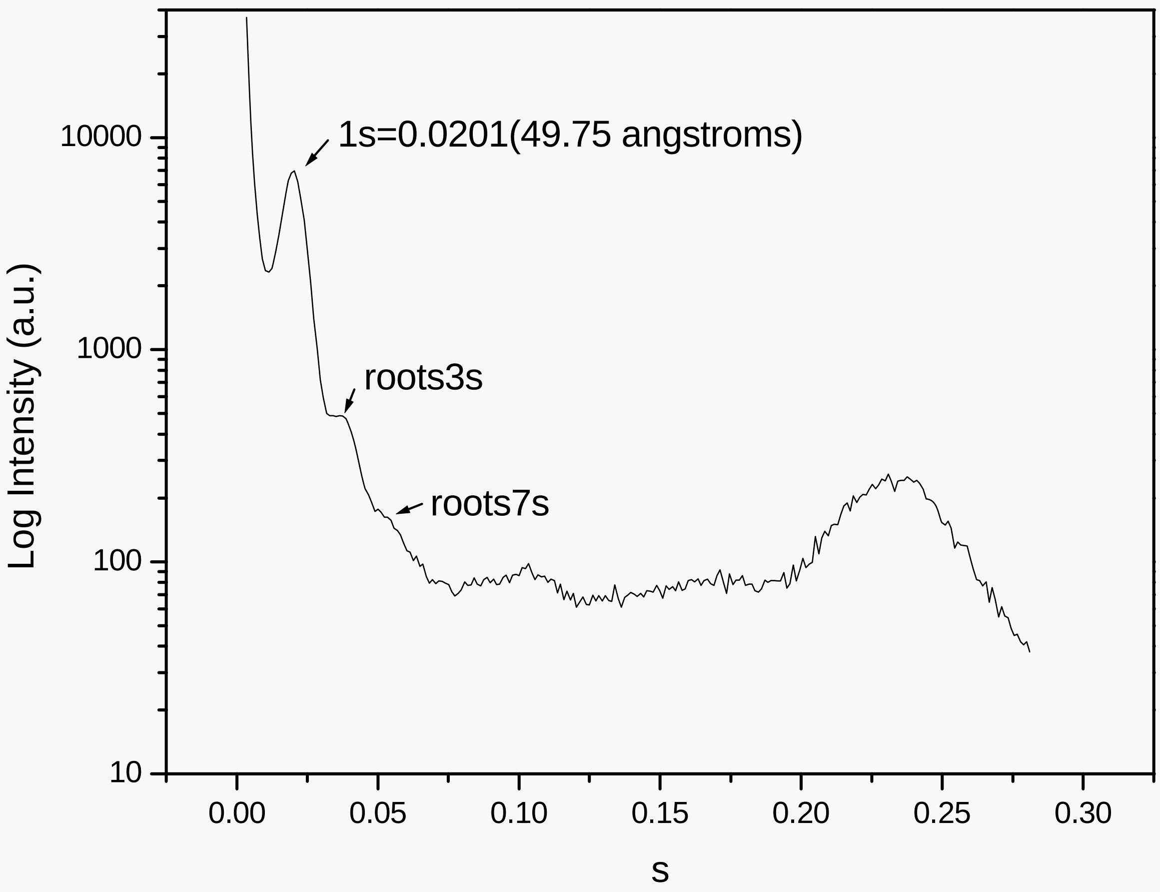


Lamella



1S=0.0217 (46 angstroms)
$s = \frac{2\sin\theta}{\lambda}$
2s
3s
4s
5s
6s
Log Intensity (a.u.)
10000
1000
100
10
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
s

Hexagonal cylinder

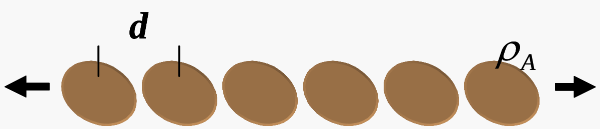


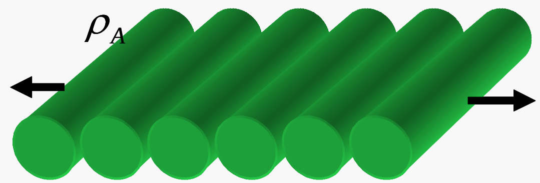
Mathematical treatment of X-ray scattering

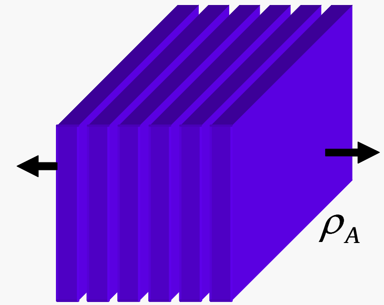
$$I(s)=\left|\Im\{\rho(x)\}\right|^2$$

$$s=\frac{2\pi\sin\theta}{\lambda}=\frac{1}{d}$$

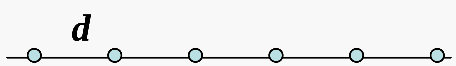
$$\rho\,(x)=\rho_A(x)*Z(x)$$
$$\Im\{\rho\,(x)\}=\Im\{\rho_A(x)*Z(x)\}$$
$$=\Im\{\rho_A(x)\}\cdot\Im\{Z(x)\}$$
$$=F(S)\cdot Z(S)$$



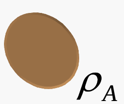




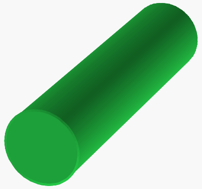
$$Z(x)\equiv\sum_{n=-\infty}^{+\infty}\delta(x-nd)$$



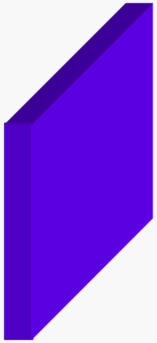
$\rho\,(x)=Z(x)*$

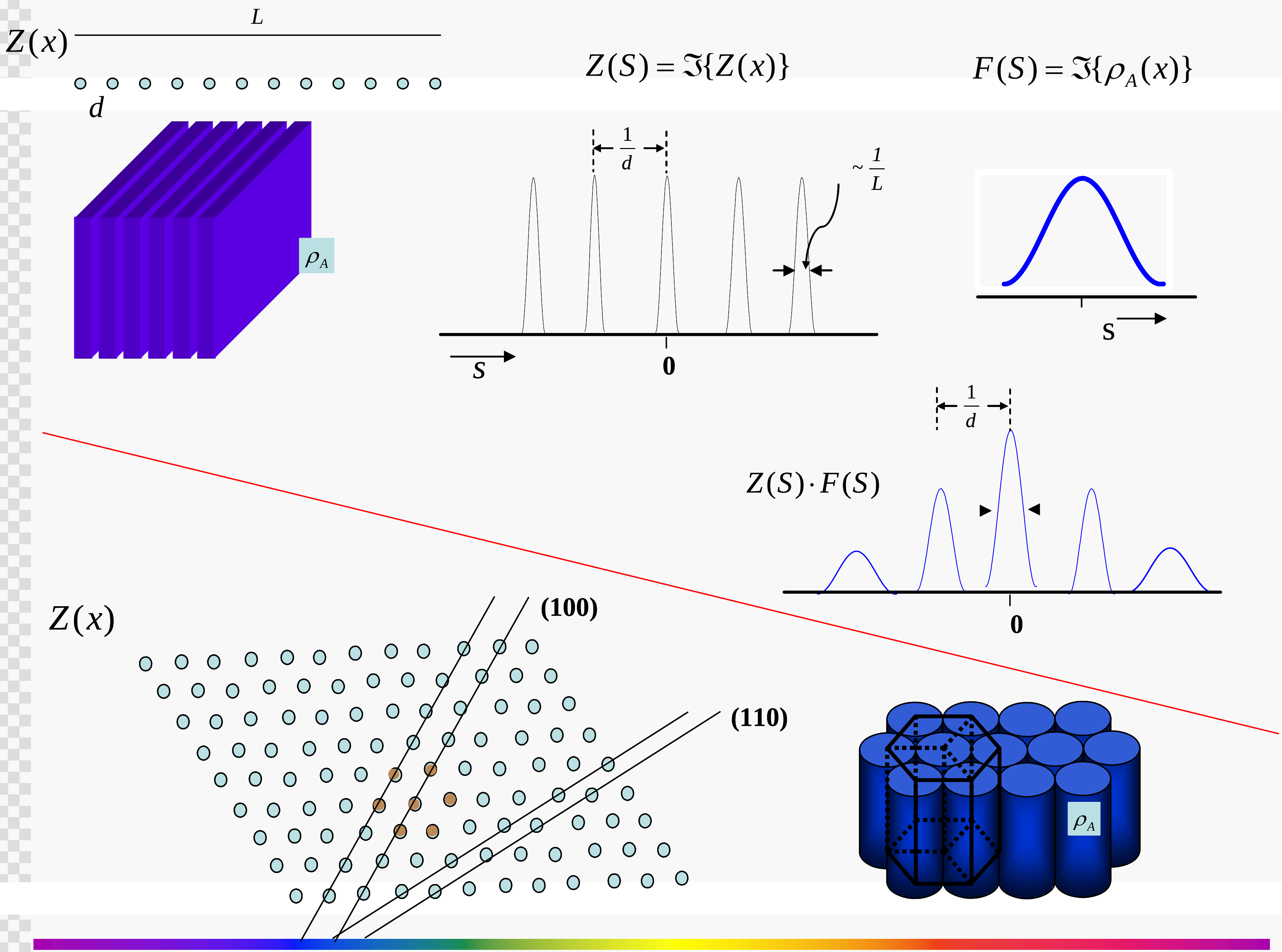


$Z(x)*$



$Z(x)*$



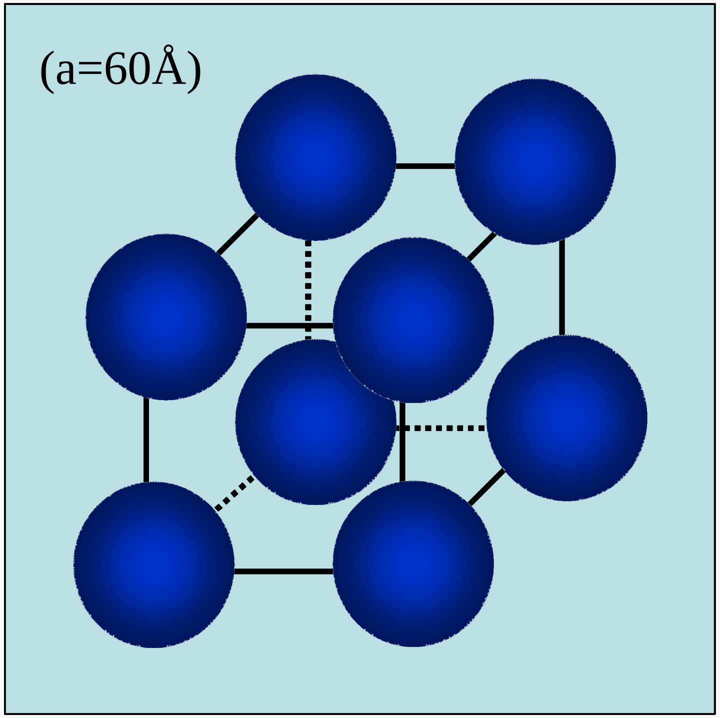


$Z(x)$
L
d
ρ_A
$Z(S) = \Im\{Z(x)\}$
$\frac{1}{d}$
$\sim \frac{1}{L}$
S
0
$F(S) = \Im\{\rho_A(x)\}$
S
$Z(S) \cdot F(S)$
$\frac{1}{d}$
0
$Z(x)$
(100)
(110)
ρ_A

Sphere cubic packing

h	k	l	s^2	s	order	d
1	0	0	2.78E-04	0.016667	1	60.00
1	1	0	5.56E-04	0.02357	√2	42.43
1	1	1	8.33E-04	0.028868	√3	34.64
2	0	0	0.00111	0.033333	√4	30.00
2	1	0	0.00139	0.037268	√5	26.83
2	1	1	0.00167	0.040825	√6	24.49
2	2	0	0.00222	0.04714	√8	21.21
2	2	1	0.0025	0.05	√9	20.00
2	2	2	0.00333	0.057735	√12	17.32
3	0	0	0.0025	0.05	√9	20.00
3	1	0	0.00278	0.052705	√10	18.97
3	1	1	0.00306	0.055277	√11	18.09
3	2	0	0.00361	0.060093	√13	16.64
3	2	1	0.00389	0.062361	√14	16.04

$$\frac{1}{d^2} = \frac{h^2 + k^2 + l^2}{a^2}$$

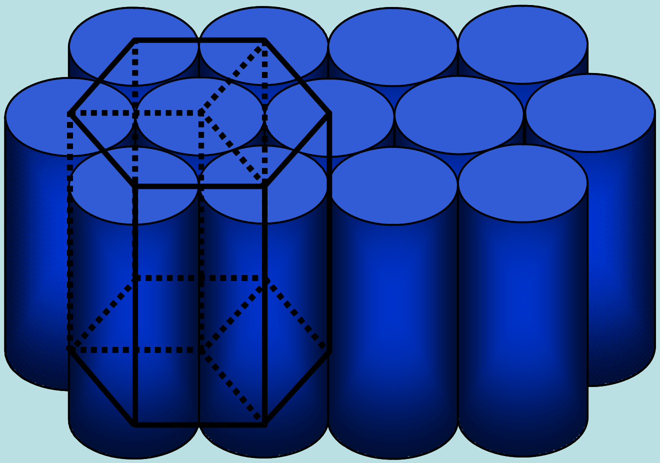


Columnar hexagonal packing

h	k	l	s^2	s	order	d
1	0	0	0.00037	0.019245	1	51.96
1	1	0	0.001111	0.033333	√3	30.00
2	0	0	0.001481	0.03849	√4	25.98
2	1	0	0.002593	0.050918	√7	19.64
2	2	0	0.004444	0.066667	√12	15.00
3	0	0	0.003333	0.057735	√9	17.32
3	1	0	0.004815	0.069389	√13	14.41
3	2	0	0.007037	0.083887	√19	11.92
3	3	0	0.01	0.1	√27	10.00
4	0	0	0.005926	0.07698	√16	12.99
4	1	0	0.007778	0.088192	√21	11.34
4	2	0	0.01037	0.101835	√28	9.82
4	3	0	0.013704	0.117063	√37	8.54
4	4	0	0.017778	0.133333	√48	7.50

$$\frac{1}{d^2} = \frac{4}{3}\left(\frac{h^2 + hk + k^2}{a^2}\right) + \frac{l^2}{c^2}$$

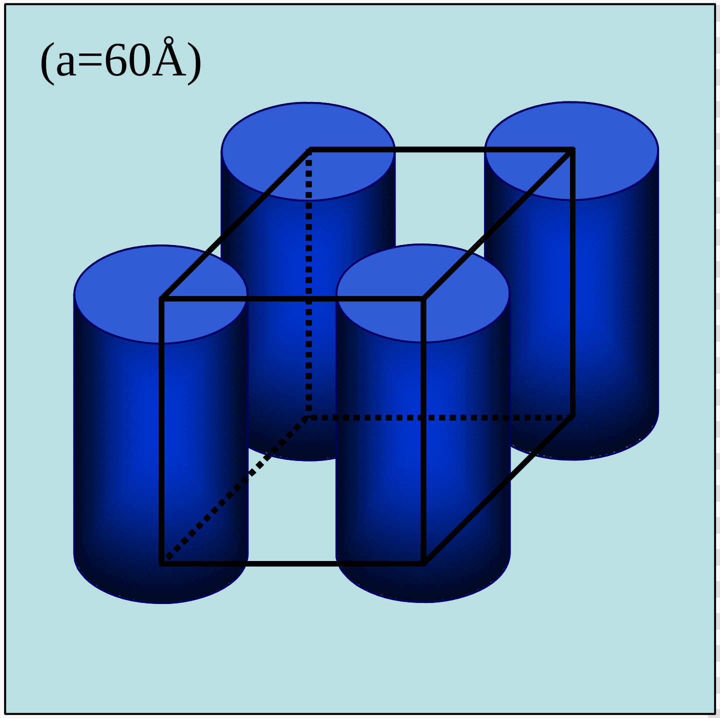
(a=60Å)



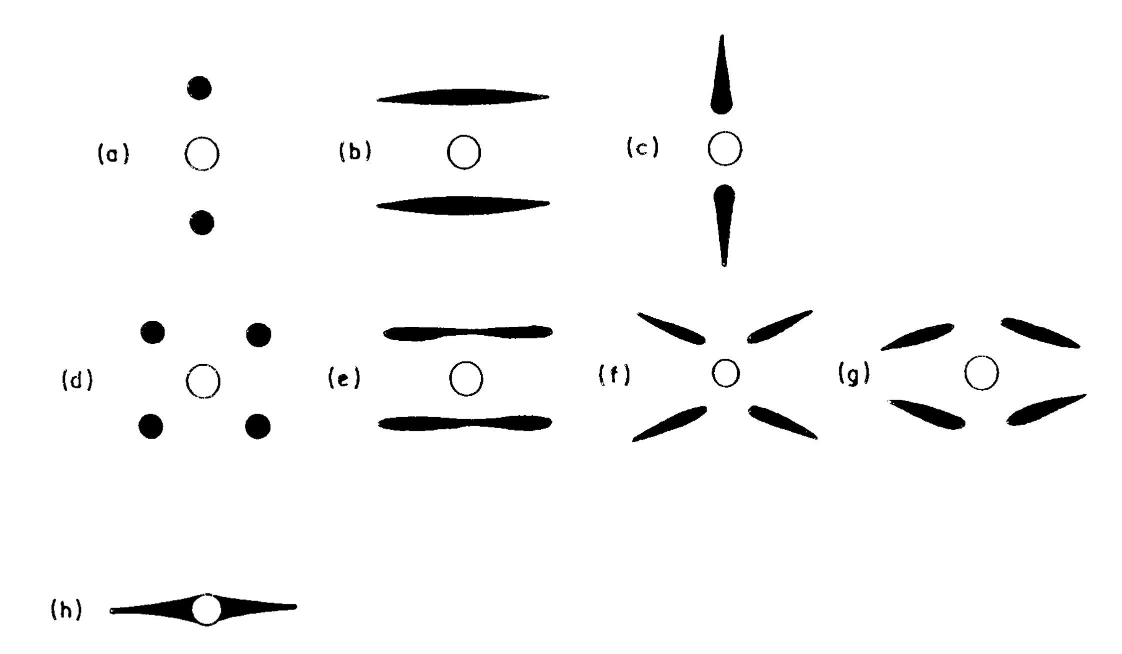
Columnar quadratic packing

h	k	l	s^2	s	order	d
1	0	0	0.000278	0.016667	1	60.00
1	1	0	0.000556	0.02357	√2	42.43
2	0	0	0.001111	0.033333	√4	30.00
2	1	0	0.001389	0.037268	√5	26.83
2	2	0	0.002222	0.04714	√8	21.21
3	0	0	0.0025	0.05	√9	20.00
3	1	0	0.002778	0.052705	√10	18.97
3	2	0	0.003611	0.060093	√13	16.64
3	3	0	0.005	0.070711	√18	14.14
4	0	0	0.004444	0.066667	√16	15.00
4	1	0	0.004722	0.068718	√17	14.55
4	2	0	0.005556	0.074536	√20	13.42
4	3	0	0.006944	0.083333	√25	12.00
4	4	0	0.008889	0.094281	√32	10.61

$$\frac{1}{d^2} = \frac{h^2 + k^2 + l^2}{a^2}$$

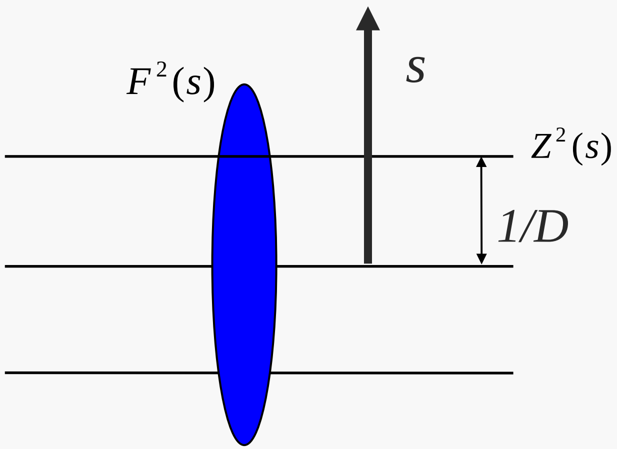


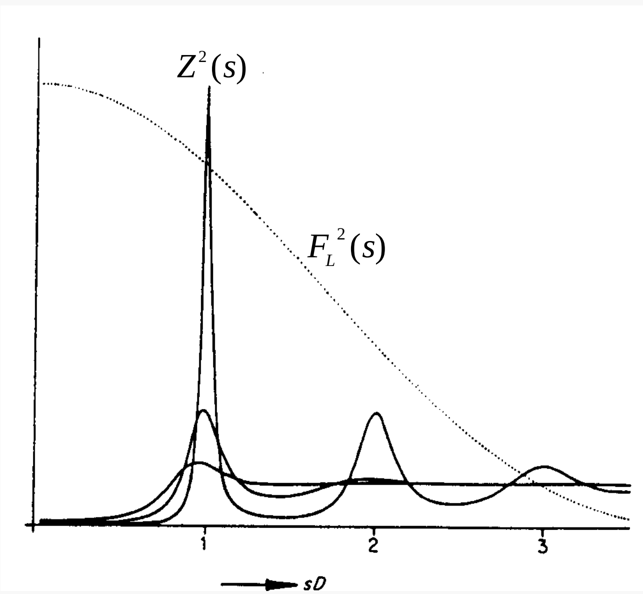
Oriented Lamellar Patterns

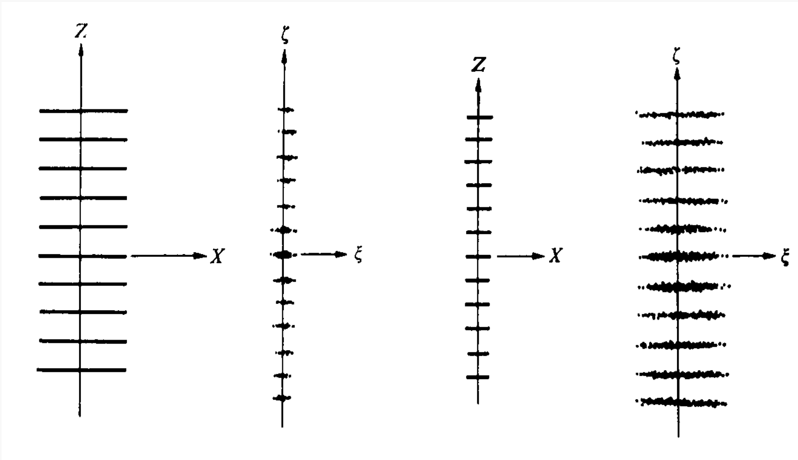


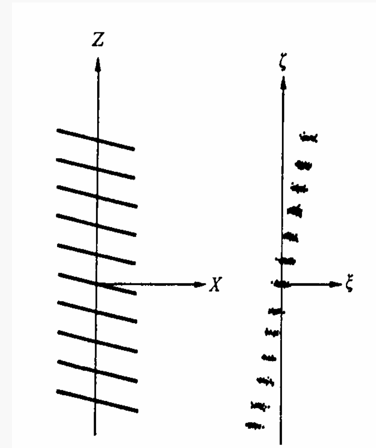
$$\rho\ (x) = \rho_A(x) * Z(x)$$
$$\Im\{\rho\ (x)\} = \Im\{\rho_A(x) * Z(x)\}$$
$$= \Im\{\rho_A(x)\}\ \Im\{Z(x)\}$$
$$= F(S)\ \ Z(S)$$

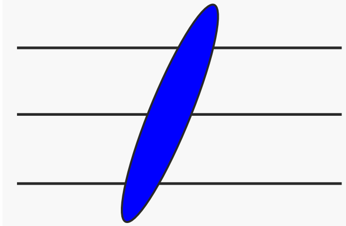
$$I_{obs}(s) = \left(F^2(s) \cdot Z^2(s)\right)$$

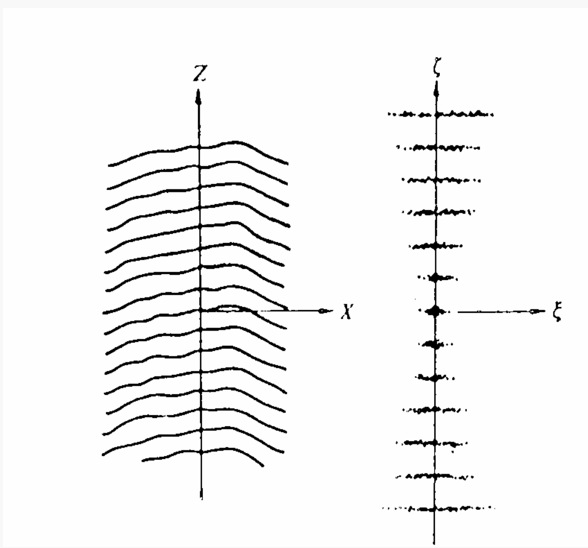


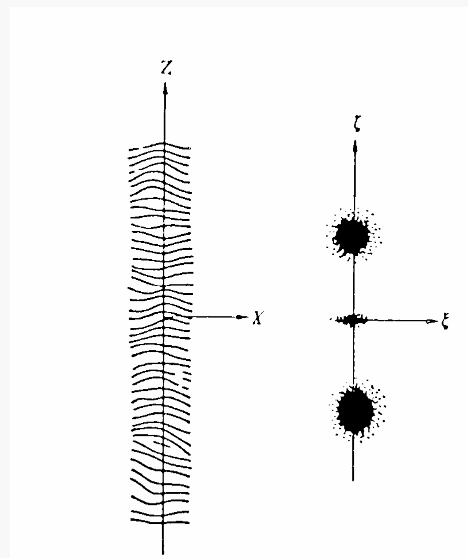



Z
X
ζ
ξ
Z
X
ζ
ξ


Z
X
ζ
ξ




Z
X
ζ
ξ


Z
X
ζ
ξ

Density Fluctuation

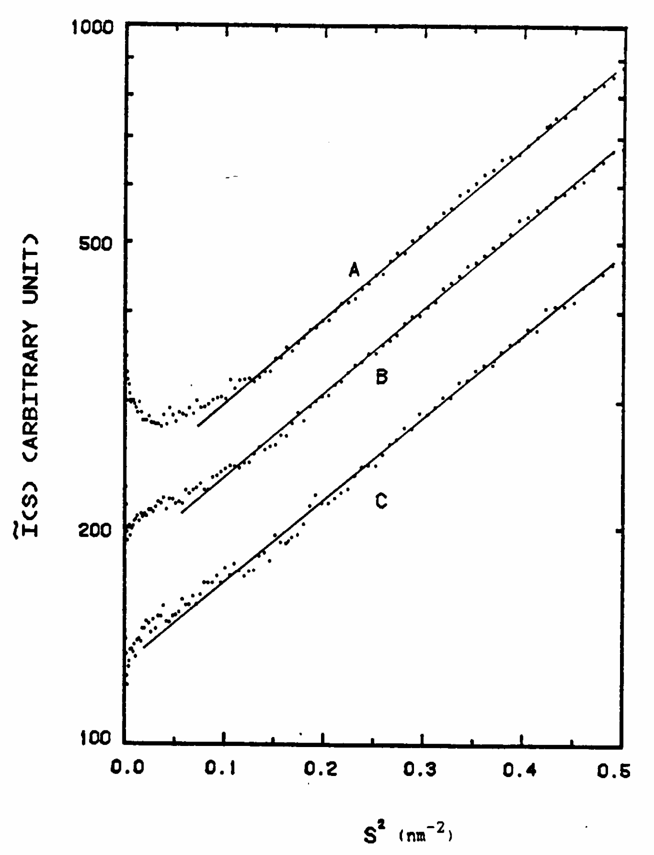
$$Fl(V) = \frac{\left\langle (N - <N>)^2 \right\rangle}{<N>}$$

$$Fl(V) = \int_{Vr} \frac{1}{\rho_0} I(s) \frac{1}{V} (\Sigma^2(s)) ds$$

Ruland (1975)

$$Fl(\infty) = \frac{1}{\rho_0} I(0)$$

$$Fl(\infty) = \rho \kappa T \beta_T$$



Optimization of Collection Time (Error Analysis)

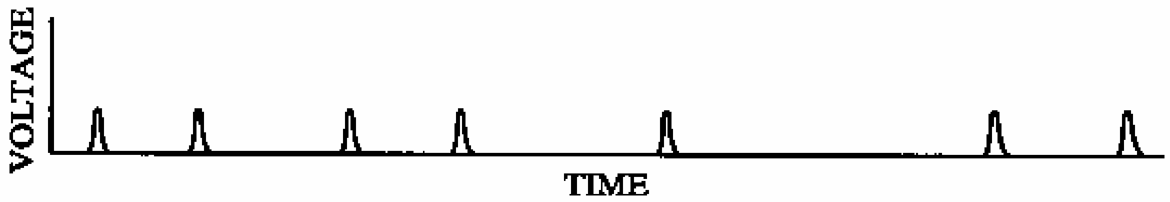


Figure 6-12 Randomly spaced voltage pulses produced by a detector.

Poisson distribution

$$P(N) = \frac{(nt)^N e^{-nt}}{N!}$$

nt: average value

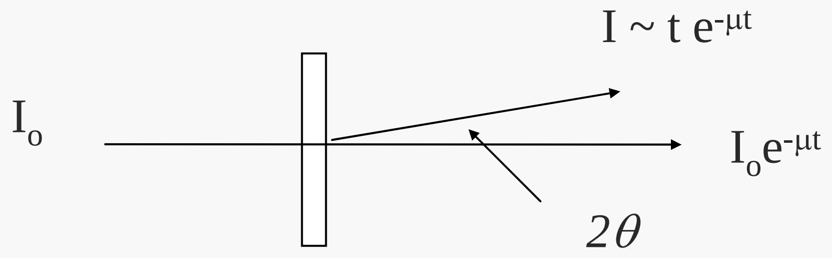
P(N) ; probability of having N count in a given time t

$$\pm \frac{\sqrt{N}}{N}$$

Relative error possessed in the count N

number of pulses counted	standard deviation (%)	collection time (sec)
1,000	3.2	1
10,000	1.0	10
100,000	0.3	100

Optimum Sample Thickness (Transmission Geometry)



t, μ : Thickness, Linear Absorption Coefficient

$$I_{obs}(s) \sim t \cdot e^{-\mu t}$$

$$t_{opt} = \frac{1}{\mu}$$